Special Issue

Reflecting on the Legacy of C.I. Lewis: Contemporary and Historical Perspectives on Modal Logic

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Guest Editors

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EDITORIAL

Preface

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We are very glad for the opportunity of dedicating the present issue of *Organon F* to modal logic, a subject that can be said to lie at the intersection of philosophy, mathematics and computer science. Nowadays there are innumerable applications of modal logic in various specific fields and it would be highly challenging to provide a detailed and all-encompassing report. The present issue is rather intended to represent a pause of reflection: it provides fundamental information on the history of modal logic from the early works of C.I. Lewis and draws attention to some current directions of research. This is the reason why we chose "Reflecting on the legacy of C.I. Lewis: Contemporary and Historical Perspectives on Modal Logic" as a title. We hope that the contents will be useful not only for readers already acquainted with the subject, but also for the broad philosophical audience interested in the formal analysis of concepts.

As the second half of the twentieth and the first two decades of the twenty-first century testify, modal logic is not simply a mathematical tool to structure our argumentations in a rigorous fashion. It is actually a way to philosophize, a multi-use framework to represent philosophical problems, and an overwhelming vocabulary to capture our basic intuitions about knowledge, meaning, reference, ethics, and, obviously, modal matters. With such a strong and determinate field comes a long and far-reaching history. The history of modal logic is indeed a subtle and complex story. Many logicians and philosophers

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were in the foreground for many years; they initiated long-lasting debates, introduced novel concepts, axioms, and techniques, or just pursued alternative philosophical lines that are still there to be examined.

Just to take one example, consider G. Bergmann. His polemical "The Philosophical Significance of Modal Logic" (1960) is quite similar to W.V.O. Quine's project to critically examine the modal notions against the background of our technical and philosophical commitments. Though Bergmann's paper is still cited sometimes, he is quite unknown in the story, even despite the fact that both of Bergmann and Quine forged their mature views at the same time and took similar examples; J. Hintikka (1963/1969) even claimed that Bergmann and Quine insisted on very similar questions. Bergmann published technical papers (1949; 1956) as well as philosophical inquiries about the modalities (1948a; 1948b; 1948c), but no members of his 'Iowa-school' took his line of thinking any further in these matters.

The history of modal logic is usually focused on its modern initiator, namely on C.I. Lewis and on his dissatisfaction with B. Russell and A.N. Whitehead's *Principia Mathematica*. Lewis thought that the way Russell and Whitehead handled the notion of 'implication' in their book was highly misleading. In *Principia Mathematica* implication is expressed through a truth-functional conditional (called 'material implication') which we can represent as $\supset$ and is such that $p \supset q$ turns out to be always true except when p is true and q is false. Thus, according to such a representation of implication, a true proposition is implied by any (true or false) proposition and a false proposition implies any (true or false) proposition. Lewis decided to look for alternative ways of capturing the intuitive meaning of 'implication' in order to avoid similar paradoxes. Though he started working on these ideas right after the publication of Russell and Whitehead's *magnum opus*, his results were published in his own philosophical-logical *epos*, *A Survey of Symbolic Logic* (1918), and later elaborated in more details with C.H. Langford in *Symbolic Logic* (1932).

After the introduction of a new operator for 'strict implication' ($\prec$), which conveys a necessary connection between the antecedent and the consequent, Lewis faced various philosophical and technical obstacles. (Note that from Lewis' first book on the issue only a few pages, not more than fifty out of four hundred, concerned the problem of the modalities.) Lewis provided much material for logicians and philosophers, and his reception was quite dynamical; sometimes philosophers got the louder voice over logicians, sometimes it was the other way around. Lewis' influence can be measured in various ways, but it might be enough to point out that even in the 1950s, years after R. Carnap's major work on the modalities, most of the logic journals (like *The Journal of Symbolic Logic*, the *Notre Dame Journal of Formal Logic*, the *Bulletin of the American Mathematical Society* or *Studia Logica*) were filled with articles that either criticized or tried to improve Lewis' work. Things began to change only after Kripke's influential paper (1959) paving the road for a semantic perspective on modal logic, though it took again a few more years till Lewis entirely left the canon. But he is back now, both as a philosopher (Olan and Sachs 2017) and as a logician.

Lewis was not the first logician who considered a formal approach to modal logic (H. MacColl and even B. Russell spent some of their time on this issue), but it was certainly Lewis who put modal logic back on the table for generations and who developed such systems for the modal notions that kept busy hundreds of philosophers, logicians and mathematicians in the forthcoming century. We believe that his historical background, philosophical influence, and logical complexity have earned the respect and attention they deserve. Even after one hundred years from the publication of *A Survey of Symbolic Logic*, there are still issues, concepts and problems that Lewis could offer us —historically, logically, and philosophically as well.

As far as the structure of the present issue is concerned, the reader will find eight research articles, that can be arranged in two groups. The first group consists of the following contributions: "Modal Logic before Kripke" by Max Cresswell, "Semantics without Toil? Brady and Rush meet Halldén" by Lloyd Humberstone, "C.I. Lewis, E.J. Nelson, and the Modern Origins of Connexive Logic" by Edwin Mares and Francesco Paoli, and "Alternative Axiomatizations of the Conditional System VC" by Claudio Pizzi. These articles were submitted upon invitation and were reviewed by the guest-editors together with external experts. The article by Cresswell provides a concise overview of the main results obtained in modal logic before S. Kripke's work on relational semantics; among other things, it points out some aspects of the early development of modal logic that are quite often overlooked in historical introductions to the topic, such as the completeness results due to A. Bayart. The article by Humberstone shows that Halldén incompleteness represents a fundamental obstacle for any attempt to obtain systematic (and trivial) characterization results in modal logic via a homophonic semantics where the metalanguage includes a copy of the logical vocabulary of the object language simulating its behaviour. The article by Mares and Paoli illustrates E.J. Nelson's early work on connexive logic, which was motivated by the desires of avoiding some paradoxes of strict implication and of developing a more fine-grained analysis of the notion of entailment; furthermore, it discusses C.I. Lewis' reaction to such an approach. Finally, the article by Pizzi provides some new axiomatic bases for a system of conditional logic originally proposed by D. Lewis as his favourite choice for the analysis of counterfactuals; furthermore, it points out some problems related to the trivialization of modalities in the logical framework at issue.

The second group consists of the following contributions: "On a Supposed Puzzle concerning Modality and Existence" By Thomas Atkinson, Daniel Hill and Stephen McLeod, "Wiredu contra Lewis on the Right Modal Logic" by David Martens, "On 'actually' vs 'dthat': Truth-conditional Differences in Possible Worlds Semantics" by Genoveva Martí and José Martínez-Fernández, and "Semantic Tableau Versions of Some Normal Modal Systems with Propositional Quantifiers" by Daniel Rönnedal. These articles were selected out of all submissions received via a double-blind reviewing procedure. The article by Atkinson, Hill and McLeod provides a novel solution to a philosophical puzzle concerning the notions of necessity, possibility and existence originally formulated by K. Fine. The article by Martens critically reconstructs K. Wiredu's thesis that the 'right' system for the formal analysis of modal notions should be at least as strong as S4. The article by Martí and Martínez-Fernández shows how possible worlds semantics can be used to rigorously capture some difference in meaning between the operators 'actually' and 'dthat'. Finally, the article by Rönnedal proposes a modular approach to build labelled tableaux for a family of normal modal systems based on a language with propositional quantification.

We would like to express our gratitude to Marián Zouhar and Martin Vacek for giving us the possibility of working on this special issue; moreover, we would like to acknowledge all reviewers and external experts for their substantial contribution in improving the overall quality of the accepted articles.[1]

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RESEARCH ARTICLE

Modal Logic before Kripke

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Abstract: 100 years ago C.I. Lewis published *A Survey of Symbolic Logic*, which included an axiom system for a notion of implication which was 'stricter' than that found in Whitehead and Russell's *Principia Mathematica*. As far as I can tell little notice was taken of this until 1930 when Oskar Becker provided some additional axioms which led Lewis in *Symbolic Logic* (written with C.H. Langford, 1932) to revise the system he had produced in 1918, and list five systems which could be obtained using Becker's suggested formulae. The present paper reviews the development of modal logic both before and after 1932, up to 1959 looking at, among other work, Becker's 1930 article and Robert Feys's articles in 1937 and 1950. I will then make some comments on the completeness results for **S5** found in Bayart and Kripke in 1959; and I will finally look at how modal logic reached New Zealand in the early 1950s in the work of Arthur Prior.

Keywords: Modal logic; history of modal logic; C. I. Lewis; strict implication; Saul Kripke

1 Introduction

In recent times modal logic has been seen as the logic of relational frames, a development which took place in the early 1960s. In order to reinforce the

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importance of this change it is timely to reflect on what modal logic was like up to that time. Such reflections are especially appropriate in that 2018 marks the centenary of the publication of C.I. Lewis's *Survey of Symbolic Logic* which represents the first presentation of a formal axiomatic system of modal logic.

2 *Principia Mathematica* (PM) 1910

Although modal logic was studied since the time of Aristotle, and was revived in the late 19th century by Hugh McColl, the modern development of the subject really begins with C.I. Lewis's dissatisfaction with some of the 'paradoxical' sentences found in Whitehead and Russell 1910. These include:

(1) $p \supset (q \supset p)$

(2) $\sim p \supset (p \supset q)$

(3) $(p \supset q) \vee (q \supset p)$

(1) and (2) can be read as claiming that if p is true then it is implied by every proposition, and if it is false it implies every proposition. (3) claims that of any two propositions one implies the other. Beginning in 1912 C.I. Lewis objected that these theorems conflict with the interpretation of $\supset$ as 'implies'.

3 C.I. Lewis *Mind* 1912, 1914, *Journal of Philosophy*, 1913, 1914

Lewis makes a distinction between an implication which holds materially and one which holds necessarily or strictly. The first article in *Mind* (Lewis 1912), after exhibiting (1)-(3) as what Lewis regards as defects in a theory of implication, concentrates on pointing out that you can understand implication as $\sim p \vee q$ provided that you understand the disjunction intensionally i.e. as holding necessarily. (Lewis 1912, 523) On p. 526 he calls the implication used in the "Algebra of logic"[1] material' and contrasts it with his own 'inferential'

[1]In footnote 1 on p. 522 he instances PM as 'the most economical development of the calculus of propositions' in particular with its definition of $p \supset q$ as $\sim p \vee q$.

or "strict" implication.' In the second of the articles in *Mind* (Lewis 1914a) he introduces two symbols for intensional ($\vee$) and extensional ($+$) disjunction respectively, though he only uses one symbol for implication ($\supset$), insisting on its ambiguity. Although in the 1912 paper he speaks of an intensional disjunction as one which is necessarily true, he does not adopt any symbol for necessity. (If $\vee$ is intensional you can of course define the necessity of p as $p \vee p$, but Lewis does not do so.) In neither of these articles does Lewis attempt anything like an axiomatisation of the logic of strict implication, though on p. 243 of (Lewis 1914a) he does provide a list of formulae which are valid, using $\supset$ for material implication and $+$ for material disjunction; and contrasting these formulae with some corresponding formulae in a language in which $\supset$ is strict implication and both $\vee$ and $+$ appear. The two articles from the *Journal of Philosophy* (Lewis 1913, 1914b) have a slightly different focus. The 1913 article lists a large number (35) of valid formulae in PM which Lewis regards as questionable when $\supset$ is interpreted as implication. As in his other articles Lewis uses $\supset$ for strict implication and $\vee$ for strict disjunction. As in (Lewis 1914a), in (Lewis 1914b) he uses $+$ for material disjunction, but in this article he introduces $<$ for material implication. He also uses $\sim$ for impossibility and $-$ for negation. On p. 591 of (Lewis 1914b) he does produce an axiom set, though two of the axioms are defective when $\supset$ is understood intensionally. One is $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$, which, when added to the other axioms, reduces each strict operator to its material counterpart.[2] It's clear that what Lewis is trying to do is retain as much of PM as he can without running into what he thinks of as its paradoxical consequences. He clearly thinks that material implication does not reflect what he supposes ordinary logicians (uncorrupted by truth-functional logic) think of as the relation of implication.

4 Lewis 1918

(Lewis 1918) is the first axiomatic presentation of modern modal logic.[3] What

[2]Problems like this are discussed in (Parry 1968). Parry notes on p. 126 that in the 1912 paper Lewis accepts the equivalence of $(p \vee (q \vee r))$ and $(q \vee (p \vee r))$ even when $\vee$ is a strict disjunction, and comments in footnote 39: "These mistakes, long since corrected, show the fallibility of logical intuition." Parry's chapter provides a thorough survey of Lewis's modal logic, and for that reason the present paper concentrates on the work of others at that time, in particular works which are not available, or not easily accessible in English.

[3]For an elaboration of this claim see (Parry 1968).

Lewis says is "Various studies toward this system have appeared in *Mind* and the *Journal of Philosophy*. ... But the complete system has not previously been printed." (Lewis 1918, 291, fn. 1).

The chapter on strict implication ($\prec$) is only a small part of the book. In that chapter Lewis does not do any metalogic, and even definitions seem to be regarded as object-language formulae.[4]

(Lewis 1918) takes impossibility ($\sim$) as primitive, and defines $p \prec q$ to mean that it is impossible that p should be true without q's being true too. In 1918 $-$ is still used for 'not' and $\sim$ for impossibility, so $- \sim p$ means that p is possible and $\sim -p$ means that p is necessary. Lewis uses juxtaposition for conjunction, but I have used the now common $\wedge$ and therefore write Lewis's definition as $p \prec q =_{def} \sim(p \wedge -q)$.[5] What you then find is a collection of axioms (Lewis 1918, 291ff.) of which the first five are no more than strict but conjunctive versions of axioms of PM. Specifically Lewis's axioms are:[6]

1.1 $(p \wedge q) \prec (q \wedge p)$

1.2 $(q \wedge p) \prec p$

1.3 $p \prec (p \wedge p)$

1.4 $(p \wedge (q \wedge r)) \prec (q \wedge (p \wedge r))$

1.5 $p \prec - - p$[7]

1.6 $((p \prec q) \wedge (q \prec r)) \prec (p \prec r)$

1.7 $\sim p \prec -p$

[4] A criticism of these early papers is found in (Wiener 1916). Although Wiener does not put the point in exactly this way his defence of Russell against Lewis could easily be seen as pointing out that when we are giving an account in the metalanguage of implication we are claiming that for any formulae α and β, α implies β when $\alpha \supset \beta$ is valid, which is to say that it is true no matter what formulae α and β are. Or, if you prefer, either α is false or β is true, no matter what values are given to its propositional variables. Lewis seems unaware of this way of describing the situation. Here and elsewhere in the paper I use α, β etc. as metalogical variables for well-formed formulae (wff) of the relevant object language. The need for making such a distinction seems not to have been recognised by these earlier writers.

[5] In doing so it must be remembered that on p. 293 of (Lewis 1918), in 1.04, Lewis uses the symbol $\wedge$, not for conjunction, but for strict disjunction. In 1.05, he uses + for material disjunction.

[6] In addition to using $\wedge$ for conjunction in place of Lewis's juxtaposition, I have inserted parentheses.

[7] This axiom is repeated in the list on p. 493 of (Lewis and Langford 1932), but was proved to be redundant in (McKinsey 1934). Lewis notes McKinsey's result in the 1959 Appendix (Appendix III) to (Lewis and Langford 1932) on p. 503.

1.8 $(p \prec q) = (\sim q \prec \sim p)$

It's not difficult to see how these axioms arose. The idea is that PM's theorems also hold when the main operator is strict, in the sense that, for instance, when $\supset$ is replaced by $\prec$ the result will still be valid.[8] 1.6 states the transitivity of $\prec$. 1.7 says that impossibility implies falsity. 1.8 is in many ways the most interesting. In the first place it was shewn by E.L. Post to lead to $\sim p = -p$, and was replaced, in Lewis 1920, by[9]

1.8* $(p \prec q) \prec (\sim q \prec \sim p)$

But even 1.8* was one which later caused Lewis some trouble. It's not difficult to see why. In the antecedent both variables are in the scope of a $\prec$ which is itself only in the scope of the main operator, while in the consequent both the variables are in the scope of $\sim$ (which for Lewis is a modal operator) which in turn is in the scope of a $\prec$ which is itself in the scope of a modal operator.

Except for Lewis's 1920 correction to 1.8 little notice appears to have been taken of his work until a 1930 article by Oskar Becker.[10]

5 Becker 1930

Although in the *Survey* Lewis had taken the one-place impossibility operator as the only modal primitive, for which he used $\sim$, his principal interest was in the defined operator $\prec$. I mentioned above that the p and q in the consequent of 1.8* are in the scope of three operators: one occurrence of $\sim$ and two occurrences of $\prec$. (Just as p in $\sim\sim p$ is in the scope of two modal operators, which prevents its elimination by a double negation rule.). In more recent notation, with $\Box$ (or L) for necessity,[11] and $\Diamond$ (or M) for possibility, Lewis's $\sim\sim p$ is equivalent to $\Box\Diamond p$, $-$ would be $\Diamond$, $\sim -$ would be $\Box$, $\sim - \sim -$ would

[8]Thus, for instance, not only is $-p \supset (p \supset q)$ valid, so is $-p \prec (p \supset q)$, but not of course $-p \prec (p \prec q)$.

[9]The label 1.8* is given by (Becker 1930, 504) (but using $<$ for strict implication). In (Lewis 1918) it is 2.2 on p. 297.

[10]There are a few exceptions. As well as (Wiener 1916) mentioned in footnote 4 above there is an article by E. J. Nelson (1930) on what have come to be called the 'paradoxes of strict implication' – that an impossible proposition strictly implies every proposition, and that a necessary proposition is implied by every proposition. (See Lewis 1918, 506: 3.52 and 3.55.) Although this topic has caused philosophical controversy I am not concerned with it in this paper.

[11]In the time in question modal notation was somewhat fluid, and, except when commenting specifically on notational matters I have used Fitch's symbol $\Box$ for necessity and Lewis's symbol $\Diamond$ for possibility. The first published use of $\Box$ is in (Barcan 1946).

be $\Box\Box$, $- \sim - \sim$ would be $\Diamond\Diamond$, and so on. These sequences of operators are what Becker called 'modalities', and in particular, where one modal operator is inside the scope of another he called them 'iterated' or 'complex' modalities. (Becker 1930, 502) comments that:

> The more complicated modalities are not handled by Lewis. It is striking that the iterations of impossibility are mentioned.[12]

By this he means that Lewis's axioms do not guarantee the reduction of iterated modalities. So (Becker 1930, 508) says:

> We therefore add to the Lewis axioms the new axiom 1.9

1.9 $- \sim p <\sim\sim p$[13]

A few pages later he introduces two further axioms:

1.91 $p <\sim\sim p$ (p. 513)

1.92 $\sim -p <\sim - \sim -p$

In current notation 1.9 is equivalent to $\Diamond p \prec \Box\Diamond p$, 1.91 is $p \prec \Box\Diamond p$, and 1.92 is $\Box p \prec \Box\Box p$. These axioms are all ones considered by (Lewis 1932). 1.91 is the proper axiom of **S5** and 1.92 of **S4**. 1.91 is the Brouwerian axiom. Becker's principles led Lewis to write the appendix to the 1932 book with Langford, in which he set out a number of modal systems of increasing strength.

6 Lewis and Langford 1932 (L&L)

I won't say much about (Lewis and Langford 1932) since it is well known, and surveys of the various Lewis systems exist all over the place. In this work Lewis takes possibility, which he writes as $\Diamond$, as the basic modal notion, and defines $p \prec q$ as $\sim \Diamond(p\wedge \sim q)$. ($\sim$ is now the regular negation, not impossibility as in Lewis 1918.) By 1932 Lewis had been convinced that even axiom 1.8* was too strong, and opted for something weaker. In an appendix Lewis sets out five systems which he calls **S1**–**S5**. The system of (Lewis 1918) is the

[12] This translation is by Jacques Riche.

[13] Becker's symbol for strict implication is $<$. I am omitting superfluous parentheses.

system he calls **S3**, and it is the presence of 1.8* in **S3** which appears to have worried him. The system he favoured was the one he called **S2**. In the 1932 book he includes a proof that **S1** is weaker than **S2**, and he is clear that **S3** is stronger than he wanted, but at the time of writing L&L he had no proof that **S3** was stronger than **S2**, and used **S1** as a 'fallback' position in case **S2** should contain **S3**. (Parry 1934 proves that **S3** is stronger than **S2**, and Parry 1939 is a study of the modality patterns of **S3**.)

Although L&L was published in 1932 the next two articles appear to have been written in ignorance of that work, and so do the papers they refer to. They mostly use the notation of (Lewis 1918), and I will follow them in this (though they sometimes use $<$ instead of $-\!\!3$). But I will refer to the systems **S3**, **S4** and **S5** even though these names were only introduced in an appendix to L&L.

7 Gödel 1933

In a one page plus three lines article, Gödel provides an axiomatisation of the system now called **S4**. Although Gödel is aware of Lewis's earlier work, and also aware of Becker's work, and of the work of W.T. Parry, he does not yet seem to be familiar with L&L. From the perspective of modal logic the main feature of Gödel's paper is to axiomatise **S4** by shortcutting the separate postulation of strict versions of the axioms of PM, and replacing them with a rule which says that if a formula is a theorem then so is that formula preceded by a necessity operator which Gödel writes as B. This rule is added to an axiomatisation of the classical (non-modal) propositional calculus, together with the axioms:

- $Bp \to p$
- $Bp \to ((B(p \to q) \to Bq)$
- $Bp \to BBp$

In these axioms $\to$ is the symbol for material implication (Whitehead and Russell's $\supset$) used in (Hilbert and Ackermann 1928).

One might wonder how Gödel came to consider an axiomatisation with such a rule. I owe the following conjecture to a discussion with Rob Goldblatt.

Gödel's paper is concerned to shew how to interpret intuitionistic logic with the aid of B as a 'provability' operator. In intuitionistic logic, mathematical truths —represented by formulae which can be 'asserted'— only exist because they have been proved. It therefore seems natural to add a rule which explicitly states this. As far as I understand, Gödel was not a modal logician, though his article does make the link with Lewis's system extended by an axiom of Becker's which is equivalent to Gödel's $Bp \rightarrow BBp$. Nevertheless, although Gödel at that stage may not have been familiar with Lewis and Langford's book he was familiar with (Becker 1930), since he had reviewed it in (Gödel 1931), and therefore he would have been familiar with (Lewis 1918), at least to the extent that Becker represents Lewis.

Gödel cites (Parry 1933)[14] as evidence that the system that he, Gödel, has just presented is indeed the system of (Lewis 1918) provided that it is extended by one of Becker's axioms. Gödel is correct that Lewis's system (**S3**) with the addition of $Np < NNp$ does axiomatise his (Gödel's) system. This axiom is Becker's 1.92 on p. 514, which gets you **S4**. (Parry 1933) however is concerned with Becker's 1.9, which is what is one form of the proper **S5** axiom. Parry's purpose is to provide a decision procedure by conjunctive normal form for first-degree modal formulae —formulae in which no modal operator is within the scope of any other modal operator. Parry does not himself prove that Becker's 1.9 guarantees that every formula can be reduced to first degree, though he does add a footnote acknowledging the appearance of (Wajsberg 1933), in which the reduction of every formula to the relevant normal form is established with the aid of 1.9. Since such a reduction is not possible in **S4**, nothing in (Parry 1933) helps (Gödel 1933). So in all likelihood it may be accidental that Gödel refers us to (Parry 1933), and probable that the connection with modal logic is incidental to Gödel's aim. It is perhaps significant that the item immediately following (Parry 1933) is a paper by Gödel, though unconnected with modal logic.[15]

8 Wajsberg 1933

Wajsberg concentrates on **S5** rather than **S4**. He uses a reduction to a normal form, and gives an interpretation in an 'extended' logic of classes which con-

[14](Parry 1933) is Gödel's Parry (1933a). Gödel's Parry (1933) is my (Parry 1934).

[15]Much of the information in this section, and indeed the whole paper, is based on material collected by Jacques Riche, to whom I am deeply indebted.

tains an operator on a class A which gives the universe class U when $A = U$ and the empty class $\emptyset$ when $A \neq U$. Wajsberg axiomatises this calculus in a way which can be seen as a set of axioms for **S5** (and he notes this). In a sense then this can be said to constitute the first completeness proof in modal logic. It does however provide completeness only for **S5**. One feature that Wajsberg's work does illustrate is that there is a difference between reducing modalities, strings of $\Box$, $\Diamond$ and $\sim$, and modal functions, formulae with modal operators in them. Though he does not use the name 1.9 or attribute the axiom to Becker Wajsberg's work shews that with Becker's 1.9 you can not only reduce all modalities to six, but you can reduce all modal functions to formulae of first degree. This contrasts with **S4** where, although there are only finitely many distinct modalities, modal functions are not always reducible to first degree.

Axiom 1.9 was first published in (Becker 1930), but from a footnote on p. 492 of L&L it seems that Wajsberg had seen its importance before Becker's work had appeared. Lewis in this footnote is commenting on some matrices which can be used to distinguish the various systems he (Lewis) is putting forward. Lewis writes:

> Groups II and III, below, were transmitted to Mr. Lewis by Dr. M. Wajsberg, of the University of Warsaw, in 1927. Dr. Wajsberg's letter also contained the first proof ever given that the System of Strict Implication is not reducible to Material Implication, as well as the outline of a system which is equivalent to that deducible from the postulates of Strict Implication with the addition of the postulate later suggested in Becker's paper and cited below as C11.[16] It is to be hoped that this and other important work of Dr. Wajsberg will be published shortly.

9 Feys 1937

In 1937 Robert Feys produced a survey of modal systems. Much of Feys's article is in the spirit of Lewis in looking at which laws of the non-modal propositional calculus still hold when the symbols are interpreted 'strictly'

[16] C11, in the notation of (Lewis and Langford 1932) is $\Diamond p \prec \sim\Diamond\sim\Diamond p$, and is the characteristic axiom of the system Lewis called **S5**. In the notation of (Lewis 1918), it is $- \sim p \prec \sim\sim p$. In Becker and in Wajsberg it is $- \sim p < \sim\sim p$.

and which do not. There is however an important difference. Feys seems to be the first to notice that you can treat modal logic as a collection of systems, all based on classical logic, but in which the modal operators can be given a variety of interpretations, and each system reflects some particular interpretation. Unlike Lewis, Feys seems to feel that it is not at all an easy matter to decide on just which logic is the 'correct' one, and points out that mainstream logic concentrates on non-modal operators. Feys sees the need to justify considering modality against those who think that 'the logic of truth and falsity is enough'. Here are some extracts from §1 of Feys's paper.[17]

> The idea of a logic of modalities is as old as logic itself.
>
> When, two thousand years after Aristotle, logistics resumes its work to give it mathematical rigor, the form it will adopt is that of a logic of truth and falsity, almost such as the Stoics had conceived it. In this restricted form it will prove susceptible of a tangible and adequate expression by symbols of which *Principia Mathematica* remains the model.
>
> But such a methodical limitation does not eliminate the problems of philosophy, and even of the sciences, which are stated in terms of modalities. It is impossible to reason about causality, about the very value of deductions, without resorting to the idea of *necessity* and to those correlative to it, the idea of *possible*, of *contingent*.

Feys then sets out the nature of the axiomatic method, contrasting the situation in modal logic with that in non-modal propositional logic which he calls 'the logic of true and false'. In the case of the logic of true and false there is another method available that is to say we can study the logic of material implication by the method of truth-tables, and therefore such logic may be held not to require the axiomatic method. By contrast, in 1937 this method seemed the only one available for modal logic.

An axiomatic presentation does not require you to attach any meaning to the formulae. Feys first lists a set of axioms known to be sufficient for the

[17] I am using a translation by Jacques Riche, but have altered Feys's own rather idiosyncratic notation to a notation with $\Diamond$ and $\Box$ for the possibility and necessity operators.

propositional calculus followed by three modal axioms. With $\Diamond$ for possibility and $\Box$ for necessity these are:[18]

- $p \supset \Diamond p$ (23.11)
- $\Box p \equiv {\sim}\Diamond{\sim}p$ (24.1)
- $(\Box(p \supset q) \wedge \Box p) \supset \Box q$ (25.3)

together with the rules:

- A logical law remains true if one substitutes for a variable p, q, r, wherever it appears, the same function containing modalities. (22.3)
- If p and $p \supset q$ are logical laws, q is a logical law.[19] (13.42)
- From a logical law p one can conclude to a logical law $\Box p$. (What is tautologically true is tautologically necessary).[20] (25.2)

In §20 Feys shows awareness of the difference between Lewis's way of proceeding, in which the axioms are all stated in terms of strict implication, and Gödel's way of proceeding:

[18]The axiom set (for classical logic) is attributed to Heyting. It takes as primitive symbols more than are customary today. In the notation of the present paper the axioms adopted are:

- $p \supset (p \vee q)$
- $p \supset (p \wedge p)$
- $(p \wedge q) \supset (q \wedge p)$
- $(p \vee q) \supset (q \vee p)$
- $(p \wedge (p \supset q)) \supset q$
- $((p \supset q) \wedge (q \supset r)) \supset (p \supset r)$
- $p \vee {\sim}p$
- $((p \supset q) \wedge (p \supset {\sim}q)) \supset {\sim}p$
- $q \supset (p \supset q)$
- ${\sim}p \supset (p \supset q)$
- $(p \supset q) \supset ((p \wedge r) \supset (q \wedge r))$
- $((p \supset q) \wedge (r \supset q)) \supset ((p \vee r) \supset q)$

[19]Feys states this rule in terms of the letters p and q. For comments on this see footnote 4 above.

[20]It is the presence of this rule which distinguishes Feys's logic from Lewis's. None of Lewis's logics weaker than **S4** contains this rule, and Lewis himself seems not to have appreciated the difference between this rule and the **S4** theorem $\Box p \supset \Box\Box p$ (Alternatively $\Diamond\Diamond p \supset \Diamond p$).

> Two methods make it possible to relate these kinds of logics to a system of postulates (it seems to us obvious to call them logics of the Aristotelian modalities). The logics of Strict Implication avoid presupposing the logic of truth and falsity; their postulates introduce, simultaneously with the true and the false, other modalities. We will follow another path, indicated in particular by Gödel. We will start from the logic of true and false and we will complete it with postulates that will introduce the other Aristotelian modalities.

Feys dropped Gödel's **S4** axiom $\Box p \supset \Box\Box p$ to give the system now called **T**. In the 1937 paper Feys seems unaware that he has produced a modal system which is none of Lewis's **S1**– **S5**.

10 McKinsey and Carnap

In 1945/46 both J.C.C McKinsey, and Rudolf Carnap produced accounts of modality which justify particular Lewis systems. (I have discussed these in Cresswell 2013.) McKinsey's characterised **S4** and Carnap's **S5**.[21] They were however not really flexible enough to provide the generality that we have come to expect for the multiplicity of systems which have been studied after the development of the relational semantics for modal logic.[22]

11 Feys 1950

In 1950 Feys explicitly presents a survey of the systems in (Lewis and Langford 1932), together with two systems $\mathbf{S1}^0$ and $\mathbf{S2}^0$, which are like **S1** and **S2** except that they lack the axiom $p \prec \Diamond p$.

As in the 1937 paper a large amount of the 1950 paper is to list the various respects in which the theorems of the various systems of strict implication do or do not match up with the theorems of the ordinary non-modal propositional calculus. In footnote 13 of this article Feys makes some remarks about

[21] Carnap's 1946 study is in fact very similar to Wajsberg's and on p. 41 in footnote 8 Carnap acknowledges his debt to Wajsberg's 1933 article.

[22] Hans Reichenbach also discussed modalities (for instance in Reichenbach 1954), but his work stands somewhat outside the development of mainstream modal logic.

the system now called **T**, which is the system presented in (Feys 1937). The footnote refers to von Wright, who mentions material from his then forthcoming (von Wright 1951b), in which the system is called **M**. As Feys notes, his system (**T**) is taken from (Gödel 1933) by omitting the **S4** axiom. Gödel's own paper only discusses a system which is (deductively equivalent to) one of Lewis's, and so may have caused Feys to miss the crucial fact that his own system is not one of Lewis's.

12 A.N. Prior

One of the clearest introductions to modal logic in the 1950s occurs in (Prior 1955). Prior was interested in modal logic from the early 1950s.[23] In the *Craft of Formal Logic*, a manuscript from 1952, Prior shews an awareness both of (Lewis and Langford 1932) and of (Feys 1950), though he is more concerned to discuss how problems which can be expressed in 'Professor Feys's notation' had already been discussed by earlier thinkers, particularly in the 18th century. So although he is becoming familiar with modal logic Prior does not yet seem to have made it his own. He does adopt Feys's symbol L for necessity, which goes well with the modal operator M, as well as the Łukasiewicz symbols for the truth-functional operators which he almost certainly got from Bochenski. He was familiar with the work of G.H. von Wright, who, in (von Wright 1951a), produces a system of deontic logic which Prior refers to in an article published in the *Australasian Journal of Philosophy* later in that same year.[24] In that article Prior appears to suggest that his own interests do not yet run to an appreciation of modern Modal/Deontic logic. His familiarity with modal logic certainly emerges in (Prior 1952) where he imagines what amounts to a two world model in which four truth values correspond to true in both, true in the first and false in the second, false in the first and true in the second, and false in both. He then points out that there will be formulae whose validity in that model depends on there being only two worlds. But perhaps Prior's strongest contribution to modal logic was in his John Locke Lectures at Oxford in 1956 (published as Prior 1957) in which he shewed how to interpret modal logic as a logic of time. For he established in that work that different assumptions about temporal ordering could lead to different systems

[23]Mary Prior stresses the importance of Prior's teacher at the University of Otago, J.N. Findlay, whom she reports as using (Lewis and Langford 1932) in a course in 1940.

[24](Prior 1951). Page references are to the reprint in (Prior 1976).

of modality. (Prior 1967, 27) speaks of a communication from Saul Kripke pointing out that **S4** was too weak to be the logic of futurity because it allowed branching futures. This insight led inexorably to the semantic study of modal logic in terms of a set of indices ordered by a relation which could satisfy different conditions. Prior returned to New Zealand after his year at Oxford in 1956, but at the end of 1958 took up a chair in Manchester.

13 Bayart 1958, 1959

In 1959 Saul Kripke produced a definition of validity, in terms of classes of models, for quantificational **S5**.[25] This was published in *The Journal of Symbolic Logic* for that year. What was less noticed was that in *Logique et Analyse* in 1958 Arnould Bayart produced a definition of validity for quantificational **S5** in which he used a set of 'possible worlds', acknowledging Leibniz but saying that, for the purposes of logic worlds could be anything at all. Kripke of course by 1963 had realised that worlds did not have to be models, as work by authors like Kanger had supposed, but (Bayart 1958) explicitly denies that they should be models. In 1959 Bayart produced a Henkin completeness proof for quantificational **S5**. An English translation of Bayart's two papers, using more familiar notation and with an historical introduction and a commentary is found in Cresswell (2015). (See also Cresswell 2016.)

14 Conclusion

Until the work of Kripke and others in the late 1950s and early 1960s modal logic was frequently faulted through not having a viable semantic theory comparable to that produced for standard propositional and predicate logic. We have seen that Feys invokes the axiomatic method in modal logic precisely to compensate for this lack. What the possible-worlds semantics does reveal is that it is the fact that modal logic has such a simple and intuitive metatheory which goes a long way to explaining just why it survived for so long even when its critics felt that it was uninterpretable.[26]

[25] It should be noted that the present survey has concerned itself solely with modal propositional logic. The early developments in modal predicate logic begin with those found in (Barcan 1946; 1947) and elsewhere.

[26] The following list of references contains all the articles referred to in the text, together with other early articles. It does not claim to be a complete bibliography of work in modal logic before the late 1950s.

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RESEARCH ARTICLE

Semantics without Toil? Brady and Rush Meet Halldén

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Abstract: The present discussion takes up an issue raised in Section 5 of Ross Brady and Penelope Rush's paper 'Four Basic Logical Issues' concerning the (claimed) triviality – in the sense of automatic availability – of soundness and completeness results for a logic in a metalanguage employing at least as much logical vocabulary as the object logic, where the metalogical behaviour of the common logical vocabulary is as in the object logic. We shall see – in Propositions 4.5–4.7 – that this triviality claim faces difficulties in the face of Halldén incompleteness, for essentially the same reasons that Halldén thought this phenomenon raised semantic difficulties for the modal logics of C. I. Lewis exhibiting it. To counter any inclination to dismiss the phenomenon as providing at best a marginal range of counterexamples to the triviality claim, a Postscript assembles some reminders of the extent of – and the varied considerations favouring – Halldén incompleteness.

Keywords: Nonclassical logic; semantic completeness; homophonic model theory; Halldén completeness; choice of meta-logic.

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1 Introduction and initial difficulties

The material under (2) of the following quotation from Section 5 (entitled "Classical meta-logic versus nonclassical meta-logic") of (Brady and Rush 2009) provides the stimulus for the present discussion; the rest of the quotation is included to provide context. Note that letters from the range $p, q, r, \ldots$ are used as sentence letters (propositional variables) while $A, B, \ldots$ are metalinguistic variables over formulas of arbitrary complexity. The bibliographical references have been adjusted to match the style used here:

> (1) Intensional logic requires an intensional semantics.
>
> This has been strongly argued by (Meyer and Routley 1977) and in the appendix of (Brady 2003), rejecting extensional reduction in favor of a semantics that more directly captures the meaning of the object logic. But, what nonclassical meta-logic is appropriate? Does this mean that it is the same logic as that of the object language? This leads us to the second point.
>
> (2) Trivial semantics (or toil-less semantics) is just that.
>
> Meyer and Sylvan (né Routley) have also made this point, in conversation. Here, such a semantics interprets each connective according to the logic of the object language, as follows:
>
> $I(p)$ is assigned T or not assigned T, for each p.
> $I({\sim}A) = T$ iff not $(I(A) = T)$.[1]
> $I(A \,\&\, B) = T$ iff $I(A) = T$ and $I(B) = T$.
> $I(A \vee B) = T$ iff $I(A) = T$ or $I(B) = T$.
> $I(A \to B) = T$ iff, if $I(A) = T$ then $I(B) = T$.
>
> A formula A is *valid* iff $I(A) = T$, for all assignments to its variables.
>
> The Soundness and Completeness Theorems are then trivial. For completeness, for a nontheorem N, $I(N) = T$ would fail as a result of the failure of the corresponding metalogical law, applied to totally unconstrained assignments to its variables.

[1] In fact what appears in (Brady and Rush 2009) at this point is "$I({\sim}A) = T$ iff not $-(I(A) = T)$", but the "$-$" is a typo (presumably intended as a hyphen linking "not" and "$(I(A) = T)$". Four lines down, italics are added here for the word "valid", since the statement is intended as a definition.

For the remainder of this section and the following two sections, we examine aspects – both technical and philosophical – of this passage and the background against which it is set. Readers keen to get to the encounter with Halldén promised in the subtitle, and described in the Abstract as threatening the above triviality claim, may like to skim hastily over that material to arrive at the start of Section 4, in which the crucial observations appear (Propositions 4.5–4.7). At the other end of the patience spectrum, those wanting to take the scenic route over the terrain of Sections 1–3 should consult the 'longer notes' on those sections collected in the Appendix as references to them appear in the main body of the paper.

In what follows, the notation "&", "$\sim$", for conjunction and negation will be used only when quoting Brady and Rush, and will otherwise be replaced by "$\wedge$" and "$\neg$". The letter "I" used by Brady and Rush is intended to suggest *interpretation*, and as we see, interpretations here are taken to be assignments of truth-values, of which at least one is T. Since this is indeed the *only* value mentioned in the above inductive definition of truth on an interpretation, there is nothing to be lost in replacing the functional notation with a less committal notation and writing "$I \models A$" in place of "$I(A) = T$".

Remark 1.1 Nothing to be lost…and in fact, there may be something to be gained by making this replacement: namely the avoidance of extraneous complications. Such complications include the following, which arose in the course of giving a semantics along these lines in (Humberstone 1996) – an enterprise not there regarded as trivial – for the case of $\Box$ understood as expressing something like metaphysical necessity. One issue here is that if identities hold of necessity if at all, then stipulating that $I(\Box A) = T$ if and only if necessarily $I(A) = T$ is in danger of committing us to: $I(\Box A) = T$ if and only if $I(A) = T$, for any I. (Instead of $I(A)$, (Humberstone 1996) has "$v(A)$" – "v" for *valuation*; see (Humberstone 1996) for a reaction to the danger. No more will be said about (Humberstone 1996) in what follows, except to point to some background clarification in note 3. The paper was written before the difficulties for homophonic model theory presented in Section 4 were noticed, and the fact that the main one stressed in that Section, connected with Halldén incompleteness, does not arise for the modal logic in play there does not guarantee that similar problems do not arise.) One might avoid this result by saying, in an informal meta-metalanguage, that "$I(\Box A)$", for a fixed I and A, is somehow not a rigid designator and so the necessity of identities does not disallow contingent identity statements in this case, or again that one rejects this principle anyway, or ... But it seems better not to get into debates about identity when the identity predicate of the object language is not even under discussion. (It must be conceded that a shadow of the worry remains after the expulsion of "=": how does it get to be contingent, for a fixed I and A, whether or not $I \models A$? Note also that it is not only in the logic of metaphysical necessity but also in the logic of vagueness that identifying the truth on an interpretation of a vague sentence with a something's being identical with something may raise difficulties for anyone thinking that identity statements can be vague – (Evans 1978) and all that.) Though we continue to use the $I(A) = T$ notation when dealing with the explicit discussion in (Brady and Rush 2009), we return to the pitfalls invited by doing so in Remark 4.4 below. ◀

Presumably it would not be appropriate to respond to the example described in Remark 1.1 by saying that a necessity operator was not one of the connectives explicitly listed in the above passage from Brady and Rush: the cases listed are there by way of example, the intention being to illustrate a general point arising with conditions for arbitrary sentence connectives; and in any case, though considerable play is made in Section 4 with modal examples, there is no special need to think of the $\Box$ figuring in them as representing any kind of necessity. Perhaps the authors intend also to include the case of quantifiers in the object language, but since their discussion is restricted to sentential logic, let us follow their lead. But we do need quantifiers in the metalanguage (and this is the reason the Abstract of the present discussion talks about the metalanguage having *at least* the logical vocabulary of the object language), since universal quantification is used tacitly over the A, B in the conditions governing $\sim$, &, $\vee$ and $\rightarrow$ and then explicitly in the definition of validity near the end of the passage.[2] Concerning the latter definition, a further corrective adjustment to the text like those alluded to in note 1 could have been made and explained there, but it raises an issue that deserves slightly greater attention than such those typographical matters. So that we don't have even the appearance of a definition in which a variable (here "I") appears free in the definiens but not the definiendum ("A is valid"), it would

[2] Adding an extra layer of complexity, we need, not only quantification over interpretations for validity but then quantification over valid formulas (or more generally, valid inferences) in the definitions of soundness and completeness – as remarked in (Williamson 2017, 159) – to say nothing of (typically) existential quantification for the notion of provability.

have been clearer to write "for all assignments I to its variables" (where this used of variables is to the propositional variables or sentence letters of the object language), but perhaps the authors did not do this because they wanted to think of I as interpreting all formulas and not just as an assignment of truth-values to the sentence letters. There are two ways of handling this issue commonly encountered. One of them makes a notational distinction between the assignment to the atomic expressions in question (here, the sentence letters), I, as it might be, and then uses the inductive definition provided by the four conditions just recalled to extend this to an interpretation function I^+, say, applying to all expressions (here, all formulas). The other is just to say that an interpretation I is any function satisfying the conditions given, and to note in passing that what I does to the atomic expressions in fact determines what it does to all expressions. On the latter approach, which I assume is not something that would raise an objection from Brady and Rush, the definition of validity would just be that a formula A is *valid* iff for all interpretations I, $I(A) = T$ (or, perhaps preferably, for the reasons given in Remark 1.1: iff for all interpretations I, $I \models A$). We return in Section 2 to what is being assumed about this "iff" in the possibly non-classical metalanguage is supposed to be.

Setting that question to one side for the moment, what we have in the quoted passage is an example of *homophonic model-theoretic* semantics – homophonic in the sense that items of the logical vocabulary of the object language – in the present case, various sentence connectives – are paired up with corresponding items from the metalanguage, which, in the case in which the metalanguage literally includes the object language, will indeed 'sound the same'. The "model-theoretic" part of the description may seem somewhat overblown for the deployment of interpretations lacking internal structure, but it enjoys a certain currency for marking the relativity of truth to something or other – models, valuations or interpretations (here I), contrasting with the case of homophonic semantics in the (absolute) truth-theoretic tradition for an interpreted language, in which there is no question of varying the interpretation concerned.[3] If in such model-theoretic semantics there is also an idle relativity to a further parameter – such as a time or a world, or more neutrally, a point in a Kripke (or similar) model – we can usefully follow the example of

[3]See (Humberstone 1996) for further discussion and references. A better name for present purposes might be *interpretational semantics*, as in (Evans 1976), but this would be distracting below, because of the use by Brady and Rush of the phrase "interpretational metalogic" whose exact significance, it will be suggested below, is at best elusive. Further references to homophonic semantics for intensional languages can be found in (Bräuner 2002, 364), though most of this is of the truth-theoretic rather than the model-theoretic kind.

(Williamson 2017) (and its precursor and companion pieces (Williamson 2011) and (Williamson 2014)) in speaking of a *quasi-homophonic* treatment in this case, as well as being more fine-grained in the application of this vocabulary to the individual clauses in the inductive definition of (interpretation-relative) truth. Thus the Kripke semantics for modal logic treats the Boolean connectives quasi-homophonically (the world parameter being idle) but the modal operators heterophonically, since they are treated, not by modal clauses in the semantic metalanguage but by (typically restricted) quantification in the metalanguage. Indeed, the introduction of the "quasi-homophonic" terminology is only called for by such semantic accounts in which at least some vocabulary is treated heterophonically, since if the ancillary parameter(s) is/are idle in every semantic clause then we can repackage them along with what we were previously thinking of as an interpretation into something and now think of this whole caboodle – for example it can be an interpretation (or model) paired with a world, to recall the case just mentioned – trading in quasi-homophonic semantic for homophonic semantics.

It should be stressed that not every attempt to use the same logic at the meta-level in discussing a certain object logic, even when the discussion concerns the model theory for that logic, is a case of homophonic model-theoretic semantics. For example, in the 1970s, W. Veldman and H. de Swart modified, respectively, Kripke models (for intuitionistic logic) and Beth trees to give structures for which (they claimed) intuitionistically acceptable completeness proofs could be provided. The references can be found in (Dummett 1977, §5.7);[4] for scepticism about such ventures, see (McCarty 2008) and references therein. A famous example from the following decade appeared in (Meyer 1985) where a relevant-logically acceptable completeness proof w.r.t. the (Routley–Meyer) heterophonic model-theoretic semantics is sought for the logic **R**.[5] More specifically, the model theory in both the intuitionistic and relevant logical cases treats conjunction and disjunction quasi-homophonically but implication and negation heterophonically. (Here in the case of relevant logic we have the 'Australian plan' in mind, with "*" function treatment of nega-

[4]This is one point at which the discussion in the first and second editions of (Dummett 1977) differ, though both writers are touched on in both editions and the difference is not pertinent to the present theme.

[5]The idea is that although this is not homophonic model theory, since the logic metalanguage is also (albeit quantified) **R** we have what we might call an 'isological' semantic treatment. It would be nicer if we could say *homological* here, but that term is already taken, though even if it weren't, we wouldn't be able to mirror the homophonic/heterophonic contrast directly since *heterological* is likewise already spoken for – in Grelling's Paradox (and opposed there to *autological*).

tion.[6] Similarly, in (Girard and Weber 2018), summarised more fully in the longer note 'Bacon and others' on Section 2 in the Appendix, the non-modal logical vocabulary is treated homophonically while the modal vocabulary is treated heterophonically (in a contraction-free metalogic). (Meyer 1985) deserves to be counted as famous, despite its never having been published (other than as an ANU research report), on the strength of the frequency which its description of the aim of the widespread practice – from which (Meyer 1985) itself departs – of using classical logic in the semantic metalanguage for relevant logic. The aim was said to be to "preach to the Gentiles in their own tongue", a description whose charm becomes especially apparent when we recall Meyer's own early days as a Christian missionary (in Japan). Among the items in the bibliography of the present paper for independent reasons in which this turn of phrase is quoted are (Read 1988) (as well as (Aberdein and Read 2009)) and (Weber, Badia and Girard 2016) – though there may be more. Read's own 'Scottish Plan' for the semantic of relevant logic (Read 1988, §7.3 and §§8.4–8.6) is, by contrast, not only metalogically relevant but treats all vocabulary homophonically.[7] (Priest 2006, 98) thinks that the use of a non-classical metalogic is not just permissible but is be mandated by considerations of integrity, writing that intuitionists and dialetheists take themselves "to be giving an account of the correct behaviour of certain logical particles. Is it to be supposed that their account of this behaviour is to be given in a way that they take to be incorrect? Clearly not. The same logic must be used

[6]See (Meyer and Martin 1986) for the contrast between this and the (Dunn–Belnap or) 'American plan' approach. Brady and Rush touch on both in their discussion but the treatment of the latter on p. 495 is somewhat garbled: we have clauses such as – to illustrate with the case of negation – $I(\sim A) = T$ iff $I(A) = F$ and $I(\sim A) = F$ iff $I(A) = T$, and it is remarked that when A and B are "classically evaluated, that is, $I(A)$ and $I(B)$ each take only one value, that is, T or F", then it "clearly follows that $\sim A$, $A \& B$, and $A \vee B$ are also classically evaluated." But the uniqueness of the value is built into the functional notation from the start, and if I is going to be a function, it should be assigning subsets rather than elements of $\{T, F\}$ to the formulas with things like "$I(A) = F$", as just seen, replaced with "$F \in I(A)$" etc. For the first-degree entailments, Belnap and Dunn dip a toe into homophonic waters in §2.5 of (Belnap and Dunn 1981), reproduced as §80.25 of (Anderson, Belnap and Dunn 1992).

[7]Semantic consequence is handled by modalizing the relevant logic, so at first blush the approach seems truth-theoretic rather than model-theoretic, in the absence of anything like quantification over interpretations, though Read thinks otherwise: first new paragraph on p. 163. Whatever the right view is of this issue, the enterprise is highly engaging. The idea is to have relevance fall out of the idea of necessary truth-preservation rather than be tacked on as an additional filter, by taking truth preservation as a matter of not having (as I shall put it) true premisses while having a false conclusion, where this "while" is understood as a metalinguistic deployment of the intensional ('fusion') rather than the extensional conjunction which relevant logic is famous for distinguishing – more widely encountered, perhaps, in subsequent years as the multiplicative rather than the additive version of conjunction from linear logic. Some further aspects of (Read 1988) were touched on in the review (Humberstone 1991).

in both ‘object theory’ and ‘metatheory’.”[8] For more on this, see the longer note ‘Dummett and others’ in the Appendix.

These heterophonic uses of non-classical logic in the metatheory of a corresponding non-classical logic are far from toil-free – or ‘toil-less’ as it is put in the quoted passage – in which Brady and Rush are certainly not suggesting that the only way to use a non-classical meta-logic is in the is this, as they say, *trivial* semantics. The gist of the qualms raised below will not be that homophonic model theory in the style presently envisaged delivers its soundness and completeness results in too trivial a way for them to have any significance, but rather than they are not automatically guaranteed to deliver the results in question, contrary to what is said in the final paragraph of the quoted passage. The main point of concern will be raised in Section 4, but immediately we shall be able to see that all is not quite right.

The problem arises in a small (because easily remedied) way already over the presentation of the homophonic model theory under (2) in the passage quoted, and in particular in its treatment of sentence letters (propositional variables). The inductive definition of $I(A) = T$ begins with a condition

$$I(p) \text{ is assigned } T \text{ or not assigned } T, \text{ for each } p$$

which reminds us that the discussion is couched in a potentially non-classical metalogic in which the law of excluded middle is accordingly not to be taken for granted but must be explicitly assumed if it is needed. An incidental correction is called for, in that it is not $I(p)$ that is “assigned” anything but p that is assigned T (or otherwise) by I, so we what is really intended is rather

$$\text{For all sentence letters } p,\ I(p) = T \text{ or not}(I(p) = T). \qquad (*)$$

But *was* there any need to impose this condition in the first place, supposing that we – or the classically educated among us – have managed to get our

[8]In another frequently cited passage from the earlier (Priest 1984, 161), Priest urged a similar line not just for the logic/metalogic distinction but for the language/metalanguage distinction: “The beauty of the paraconsistent approach to logic is that it finally renders the object-language/meta-language distinction unnecessary in any shape or form.” The frequent appeals to that distinction in our discussion might seem to make it something Priest would object to, though as long as he is prepared to distinguish between using and mentioning an expression, the hope would be that only minor reformulations here and there would be needed to render the discussion unobjectionable. Some of the discussion of Priest’s work in (Humberstone forthcoming) suffers from a failure to distinguish, as is done here, between isological semantics in general and homophonic semantic in particular, as Priest notes at the end of his reply to that contribution in the volume in which (Humberstone forthcoming) appears.

heads around seeing it as a non-vacuous? Surely not, if this is supposed to be a general treatment, since this gives an unwanted illustration of how the semantics described delivers its verdicts of validity or invalidity, in the present case of the former verdict:

Example 1.2 Let I be an arbitrary interpretation for the language with connectives as described in the passage quoted from Brady and Rush and let us consider for an arbitrary interpretation I and sentence letter p, whether or not $I(p \vee {\sim}p) = T$. The inductive clause for $\vee$ tells us that this is the case iff $I(p) = T$ or $I({\sim}p) = T$, and the clause for $\sim$ tell us the second disjunct here holds iff not$(I(p) = T)$, so presumably altogether the account tells us – and if it doesn't, it doesn't allow us to compute $I(A)$ from the various $I(B)$ for proper subformulas B of A in any instance – that $I(p) = T$ or not$(I(p) = T)$, which is what (*) says must be satisfied for an arbitrary I and p; thus what we have shown is that the formula $p \vee {\sim}p$ is valid now for any sentence letter p. (Could we say instead that $A \vee {\sim}A$ is valid for any formula A? We don't know enough about the meta-logic to whether this general form is available. If it is not, eyebrows will naturally be raised as the object logic's claim to being regarded as a logic at all; see note 30.) A more rigorous version of the reasoning here can be presented using the apparatus introduced in Section 2; in particular, we will attend to the kind of equivalence needed to legitimate the substitutivity or replacement properties invoked here. ◀

Now, Example 1.2 is bad news. Nobody said anything about the logics under discussion being restricted to those accepting the law of excluded middle, which is described in (Brady and Rush 2009, 494) as one of the two key properties of classical negation (the other being inference by disjunctive syllogism), and not one to which our authors are especially attracted: "Second, when we come to apply logic to the physical world, we discover that there is no definitive guide enabling the separation of subjects into, for example, those to which the law of excluded middle applies, and those to which it does not." (Brady and Rush 2009, 497). Further clear reluctance to endorse unrestricted appeals to excluded middle can be found in (Brady and Rush 2008). Thus the best thing to do with (*) is to pretend it was never laid down as a condition.[9]

The fact that Brady and Rush did impose that condition should give us pause over the confidence with which the quoted passage finishes:

[9]Brady and Rush may have a reply available which hangs on an obscure distinction we shall, in the following section, find them drawing between the *interpretational* metalogic and the *underlying* metalogic.

> The Soundness and Completeness Theorems are then trivial. For completeness, for a nontheorem N, $I(N) = T$ would fail as a result of the failure of the corresponding metalogical law, applied to totally unconstrained assignments to its variables.

No argument is given in connection with soundness – though the sentiment expressed seems correct and will be illustrated in Section 2 – and the treatment of completeness begins with the same disregard of free variables as we saw in the definition of validity, leaving "I" unbound in the assertion that for an arbitrary nontheorem N, $I(N) = T$ would fail: what I? No I has been mentioned! Because the definition of validity began with a universal quantifier, what we have to show is that it is not the case that for all I, $I(N) = T$. If, by contrast with intuitionistic logic say, our logic happens to treat $\neg\forall I$ and $\exists I\neg$ as equivalent, then we can formulate the task of showing completeness as that of showing the existence of *some* interpretation I for which we do not have $I(N) = T$, and in view of Example 1.2 $p \vee {\sim}p$ for a logic lacking this as a theorem (with p a sentence letter), this cannot be done – we precisely do not have all the valid formulas provable. The question of whether, with (*) discarded, there remain problems about completeness, will remain in abeyance until Section 4, where grounds for a negative response will be given – unless the question is to be reconstrued entirely (a possibility aired in Section 5). The suggestion that we know that not every interpretation verifies N "as a result of the failure of the corresponding metalogical law, applied to totally unconstrained assignments to its variables" sounds like a gesture in the direction of an argument, but can it be spelled out more precisely? We return to this question in the discussion following Example 2.1 below (as well as in Section 5). An incidental issue arising at this point discussed in the Appendix in the longer note 'Soundness without reference to a particular axiomatization'.

2 More detail

We may summarize the homophonic model-theoretic treatment of connectives by describing its clauses in the truth-definition schematically. With the "$I(A) = T$" form replaced by "$I \models A$", the general case, for n-ary connective # whose metalinguistic analogue we denote by $\underline{\#}$, would be as follows, understood as holding for all $A_1, \ldots A_n$:

$$I \models \#(A_1, \ldots, A_n) \textit{ iff } \underline{\#}(I \models A_1, \ldots, I \models A_n). \qquad (\#/\underline{\#})$$

Under (2) in the quotation from (Brady and Rush 2009) given above, "not", "&", "∨" and "if … then ___" are taken as ways of writing what in the notation deployed in (#) would be written as $\underline{\sim}$, $\underline{\&}$, $\underline{\vee}$, $\underline{\rightarrow}$.[10] Some may prefer to omit the underlining and say we are dealing with the same connective; certainly every logical principle holding for the various # of the object language must hold in the meta-logic for the corresponding $\underline{\#}$, if the discussion under (2) is taken to be an elaboration of an affirmative answer to the question under (1): "Does this mean that it is the same logic as that of the object language?". Taken literally, this affirmative answer means even more, adding the converse demand: every metalinguistic connective must also be the analogue of some object language connective. This raises the question of the status of the "iff" pervading the example given by Brady and Rush, and retained in the formulation of $(\#/\underline{\#})$.

One might think that Brady and Rush assumed that "… iff __" was to be understood as "If … then __ and if __ then … ", and thus already available as a derived connective given the "and" and "if/then" (or $\underline{\&}$ and $\underline{\rightarrow}$) appearing on the right-hand sides of the clauses of the inductive definition of $I(A)$ appearing in the quoted passage. But if the example they are describing there, with an unspecified logic governing the connectives involved in those clauses, is meant to be representative, there is the possibility of the object logic providing us without the materials to define such an "iff". In this case two options present themselves: we could either make it clear that the metalanguage may contain such additional logical vocabulary, not present in the object language, or subject the claim that 'trivial' semantics is always available, whether or not it deserves to be called trivial, to the condition that only such logics are under consideration as provide the additional expressive devices needed.

This, however, would not be what Brady and Rush have in mind, though one does have to look around a bit to see this. The discussion in (Brady and Rush 2009) immediately following that quoted in Section 1 is this (p. 505f.):

> There seems to be a paradox here, which needs resolution. In order to do this, we need to find a meta-logic which is something

[10]We do not underline metalinguistic quantifiers ∀, ∃ since for present purposes they have no analogues in the (purely sentential) object language to contrast with.

> of a compromise between mimicking the logic of the object language and using the standard classical meta-logic. Perhaps we can answer this question best by using the distinction between interpretational and underlying meta-logic, made by Brady in (Brady 2003, chapter 13), where the interpretational meta-logic interprets the connectives and quantifiers in an appropriately nonclassical way, while the underlying meta-logic interprets the logic of the meta-linguistic infrastructure supporting the interpretation of the connectives and quantifiers.

Since this follows straight on from the passage quoted in Section 1, we might pause to ask what it is from the end of that passage to say that there "seems to be a paradox here, in need of resolution." The point is presumably that it seems wrong to impose an alien (e.g., classical) metalogic on the (typically non-classical) object logic under discussion, but if we respond to this by instead using the same object logic at the metalogical level, then this trivializes the discussion by making the provision of soundness and completeness results automatic. Not exactly a paradox, perhaps, but something which if it were a problem, might be worth responding to. In Section 4 we shall see that the would-be problem does not quite arise as Brady and Rush think: such results are *not* automatically available for homophonic model-theoretic semantics. Where they are forthcoming, however, one may wish to do non-homophonic but still 'isological' model theory (to use a term from note 5). Perhaps that is what Brady's interpretational vs. underling metalogic distinction is meant to amount to. Digging a little further, we shall come around to something we can bring to bear more usefully on the earlier passage, and that concerns the significance of the "iff" that appears in it. The excavation in question is deferred to the Appendix ('Brady on the interpretational meta-logic').

Since the logical devices required for homophonic model theory of the kind described as trivial in (Brady and Rush 2009) include not only a suitable "iff" but also the universal quantification (over interpretations I) used in the definition of validity,[11] if we want to retain the restriction to purely sentential languages, aspects of the first option will be needed to deal with that, whether

[11] Since conditions of the form (#) above – or those from Brady and Rush – tacitly involve quantification over all interpretations I and formulas $A_1, \ldots, A_n$, one might discern a universal quantifier needing to be made explicit here too. However, one could conceivably take these conditions as schemata and avoid the issue. But when it comes the notion of validity, especially as this features in *antecedent* position in an implication, the universal quantification over interpretations is inescapable, as in saying that if a given formula A is valid, it is provable (a consequence of the claim that object logic is complete).

or not the same route is taken in the case of "iff". (A formula A is defined to be valid in the quotation above if "$I(A) = T$, for all assignments to its variables," which more explicitly must be take to mean "$I(A) = T$ – or "$I \models A$", on the =-free notation preferred treatment here – for all interpretations I".) Whether or not we take "iff" to be $\underline{\#}$ for some (not necessarily primitive) object language binary connective $\#$, we should pause for a moment to consider what its logical behaviour needs to be like.

The question of what is required of the logic of "iff" raises the prior question of how to conceive of logics for present purposes. Since the envisaged toil-free semantics lays down conditions governing the behaviour of object language connectives such as $\vee$, saying that $I \models A \vee B$ iff $I \models A$ or $I \models B$ (for all interpretations I, and formulas A, B), the logic had better tell us what, by its lights, follows from such conditions. Accordingly it is natural to take logics to be consequence relations, even though the remarks on soundness and completeness at the end of the quotation from (Brady and Rush 2009) above show that their primary conception of logics appears to be as sets of formulas. However, other parts of (Brady and Rush 2009), such as p. 502, to say nothing of such ventures as (Brady 1994) and (Brady 1993), show that even when setting out a logic axiomatically, they have their eye on the consequence relation induced by the axiomatization in more or less the familiar way,[12] and we shall follow suit here.

Having opted to conceive of logics as consequence relations, we can resume our discussion of the issue of "iff", and suggest that a natural way to resolve the issue is to require that we are dealing with an *equivalential* consequence relation, which is to say a consequence relation $\vdash$ whose language, L, say, provides a set of formulas $E(p, q)$ in two variables which is such that for all $A, B \in L$, we have the following, $E(A, B)$ being the set of formulas resulting from substituting A and B respectively for p and q in the formulas in $E(p, q)$:

$$E(A, B), C(A) \vdash C(B) \textit{ for all } 1\textit{-ary contexts } C(\cdot) \qquad \text{and} \qquad \vdash E(A, A)$$

[12]The familiar – or simple-minded, as it is called in (Humberstone 2010) – way of defining a consequence relation from a Hilbert system with given axioms and rules is that we take A to be a consequence of a set of formulas if A can be obtained from formulas in the set together with axioms, by applications of the rules. In (Brady and Rush 2009) we are not quite dealing with a standard Hilbert system, however, whence the description as "more or less" familiar, because of the presence, as in numerous other publications of Brady's, of what he calls meta-rules. As suggested in (Humberstone 2008, 442), these play the role of sequent-to-sequent rules in a Gentzen system, and we may define the consequence relation induced by a Hilbert system with meta-rules to be the least consequence relation closed (when considered as a collection of single-succedent sequents) under these rules and such that when A can be derived from the axioms and the formulas in a given set by the (ordinary) rules, A is a consequence of that set.

where, on the left, and the "$C(\cdot)$" notation is as explained in note 45 (in the Appendix), while, on the right what is meant is that for each formula $D \in E(A, A)$, we have $\vdash D$ (i.e., $\varnothing \vdash D$). To make life easier, we will take the additional step of assuming that $E(p, q)$ contains a single formula, to be written as $p \Leftrightarrow q$; this is done for expository convenience and is not essential to the points under discussion. While no corresponding such connective is presumed to be part of the object language's vocabulary, it may often be that there is a corresponding connective, $\leftrightarrow$, say, and we may take $\Leftrightarrow$ as $\underline{\leftrightarrow}$.

When referring to the consequence relation on the metalanguage for discussing the object-logical consequence relation $\vdash$, we will co-opt the underlining convention just recalled and denote the metalogical consequence relation by $\underline{\vdash}$. From a similar desire to make it quite clear whether it is the object language or the metalanguage that is issue, we continue to use capital Latin letters as schematic for object language formulas, but use lower case Greek letters as schematic for formulas of the metalanguage. With all these assumptions and conventions in force the above equivalentiality conditions emerge as (2.1):

$$\text{(a)}\ \underline{\vdash}\, \varphi \Leftrightarrow \varphi \qquad \text{and} \qquad \text{(b)}\ \varphi \Leftrightarrow \psi, \chi(\varphi) \,\underline{\vdash}\, \chi(\psi) \tag{2.1}$$

understood as holding for all formulas φ, ψ, χ (or $\chi(\cdot)$) of the metalanguage, with a similar understanding in the case of the equivalent set of conditions (2.2):

$$\text{(a)}\ \underline{\vdash}\, \varphi \Leftrightarrow \varphi, \quad \text{(c)}\ \varphi \Leftrightarrow \psi, \varphi \,\underline{\vdash}\, \psi, \quad \text{and} \quad \text{(d)}\ \varphi \Leftrightarrow \psi \,\underline{\vdash}\, \chi(\varphi) \Leftrightarrow \chi(\psi) \tag{2.2}$$

The (2.1) formulation has the pleasant feature of echoing for $\Leftrightarrow$ the natural deduction introduction (in the case of (a)) and elimination rules (in the case of (b)) for the identity predicate in first-order logic (a theme taken up in Humberstone 2011, 603f.), while the (2.2) formulation show the relation between having a set (here a singleton set) of equivalence formulas and having a set of implication formulas, as these notions figure in contemporary algebraic semantics: for the latter, retain (2.2a, b) and drop (2.2c), preferably at the same time re-notating "$\Leftrightarrow$" to "$\Rightarrow$" to avoid confusion.[13] Now we can be more

[13]Note that these conditions are considerably weaker than those mentioned in the Appendix (note 45) in connection with Rasiowa.

precise about the "iff" in $(\#/\underline{\#})$, repeating it here as clarified (though not re-named):

$$I \models \#(A_1, \ldots, A_n) \Leftrightarrow \underline{\#}(I \models A_1, \ldots, I \models A_n). \qquad (\#/\underline{\#})$$

The point of all this is that to cash out the interpretation-relative truth-conditions of complex formulas in terms ultimately of their atomic constituents' truth-conditions in accordance with the homophonic inductive clauses, we need to make replacements in the scopes of various connectives. To take the simplest possible case:

Example 2.1 Suppose, to give the simplest possible illustration, that we have two object-language one-place connectives $\Box$ and $\neg$ and accordingly two matching metalinguistic connectives $\underline{\Box}$ and $\underline{\neg}$, as well $\Leftrightarrow$ as the universal quantification over formulas and interpretations which is invisible in $(\#/\underline{\#})$ but taken to be present in the instances of this schema we invoke (with $n = 1$ in each case) for these connectives. We want to calculate when $I \models \Box\neg p$, say. $(\Box/\underline{\Box})$ tells us that for any I and A, we have $I \models \Box A \Leftrightarrow \underline{\Box}(I \models A)$, so taking A and $\neg p$ we have

$$I \models \Box\neg p \Leftrightarrow \underline{\Box} I \models \neg p \qquad (2.3)$$

and similarly invoking $(\Box/\underline{\Box})$ we get

$$I \models \neg p \Leftrightarrow \underline{\neg}(I \models p) \qquad (2.4)$$

Now we may instantiate the (meta-)schema (2.2d) with (2.4) on the left and taking $\chi(\cdot)$ as $\underline{\Box}(\cdot)$:

$$\big[\, I \models \neg p \Leftrightarrow \underline{\neg}\,(I \models p)\big] \;\underline{\vdash}\; \big[\underline{\Box}\, I \models \neg p \Leftrightarrow \underline{\Box}\,\underline{\neg}(I \models p)\big]$$

Here, to make it clear among a plethora of various turnstiles that the main relational symbol here is the $\underline{\vdash}$, I have put square brackets round the formulas represented as $\varphi \Leftrightarrow \psi$ and $\chi(\varphi) \Leftrightarrow \chi(\psi)$ in (2.2d). Since our semantic theory for the object logic delivers (as (2.4)) the formula on the left, the right-hand side here is a consequence, according to the metalogic $\underline{\vdash}$, of that theory, and this formula

$$\underline{\Box}\,(I \models \neg p) \Leftrightarrow \underline{\Box}\,\underline{\neg}(I \models p) \qquad (2.5)$$

has for its left-hand side the right-hand side of (2.3), and the conditions laid down for $\Leftrightarrow$ suffice to ensure that for any α, β, γ,

$$\alpha \Leftrightarrow \beta, \beta \Leftrightarrow \gamma \,\underline{\vdash}\, \alpha \Leftrightarrow \gamma,$$

so taking α, β, γ as $I \models \Box\neg p$, $\underline{\Box}(I \models \neg p)$, and $\underline{\Box}\,\underline{\neg}(I \models p)$ respectively, we get from (2.3) and (2.5), the final stage to be reached in spelling out the truth-on-I conditions of $\Box\neg p$:

$$I \models \Box\neg p \;\Leftrightarrow\; \underline{\Box}\,\underline{\neg}(I \models p),$$

which completes our illustration – laborious as it has been for such a simple point – of the reason we need a "$\Leftrightarrow$" in the metalanguage with the properties we have demanded of it (or at least a set $E(p, q)$ of equivalence formulas to play the $p \Leftrightarrow q$ role). ◀

Let us turn now to the remark with which the passage from (Brady and Rush 2009) quoted in Section 1 concluded:

> The Soundness and Completeness Theorems are then trivial. For completeness, for a nontheorem N, $I(N) = T$ would fail as a result of the failure of the corresponding metalogical law, applied to totally unconstrained assignments to its variables.

The sudden reference out of nowhere to I already attracted our attention in Section 1, but here we want to attend to the soundness side of the case. (We return to the issue of semantic completeness in Section 4, and to what Brady and Rush say about it in the passage just quoted, in Section 5.) By way of commentary on the triviality claim for the soundness result, let us take as an example of our object language the purely implicational logic *BCK*. As already noted, what we really need to have in mind is a consequence relation, certainly for meta-logical reasoning and in this case, as already noted, we will need more than just implication in our logical we will also need universal quantification, so if the object logic is given by the consequence relation $\vdash_{BCK}$ defined presently, then the metalogic will be governed by a consequence relation – $\underline{\vdash}_{BCK}$ – which mimics $\vdash_{BCK}$ propositionally but also includes quantificational and equivalential resources. Section 1 stressed also the need to include an 'iff' surrogate to handle the inductive definition of truth (on an interpretation) and $\vdash_{BCK}$ is well known to provide a set $E(p, q)$ of equivalence formulas in the shape of $\{p \to q, q \to p\}$, which, in order to stay as close to the original passage from Brady and Rush, we shall condense into a singleton presently for the case of $\underline{\vdash}_{BCK}$. Returning to $\vdash_{BCK}$ itself, this consequence relation is defined to relates a set Γ of formulas to a formula C just when C can be obtained from the formulas in Γ by successive applications of Modus Ponens (the rule taking us from $A \to B$ and A to B) to formulas which are either elements of Γ or instances of the axiom schemes

$$(B \to C) \to ((A \to B) \to (A \to C)) \qquad \text{(B)}$$
$$(A \to (B \to C)) \to (B \to (A \to C)) \qquad \text{(C)}$$
$$A \to (B \to A) \qquad \text{(K)}$$

Although in other publications Brady has concerned himself with the semantics of rules and consequence relations, the emphasis in the passage quoted from (Brady and Rush 2009) seems to be on soundness and completeness for logics as sets of formulas and thus as potentially presented by a Hilbert system – skirting around the issue raised in the longer note 'Soundness without reference to a particular axiomatization' on the preceding section (in the Appendix) – which in the current instance amounts to showing the validity (= truth on all interpretations) of, respectively, all and only those formulas which are $\vdash_{BCK}$-consequences of the empty set.

And it is clear in general outline how the recipe for a soundness proof sketched in the passage quoted is supposed to go. In relatively informal terms, taking the case of (K) as the shortest schema by way of illustration, we need to show that any instance of this schema is valid. So let us take an arbitrary interpretation I, with a view to showing that $I \models A \to (B \to A)$, A and B as in (K), now taken as standing for particular unspecified formulas rather than as general schematic letters. Let us recall the specific instance of $(\# / \underline{\#})$ $\#$ is $\to$:

$$I \models A \to B \Leftrightarrow (I \models A \underline{\to} I \models B). \qquad (\to/\underline{\to})$$

Giving the name ($\underline{\mathsf{K}}$) to the corresponding meta-logical schema, i.e. $\varphi \to (\psi \to \varphi)$, we reason as follows, using $\Leftrightarrow$ as above for convenience (though since $\{p \to q, q \to p\}$ is a set of equivalence formulas for $\vdash_{BCK}$, we don't actually need to wheel in a new connective here).

1	$I \models A \underline{\to} (I \models B \underline{\to} I \models A)$	instance of ($\underline{\mathsf{K}}$)
2	$I \models A \underline{\to} I \models B \to A$	replacing the rhs of 1 by $(\to/\underline{\to})$
3	$I \models A \to (B \to A)$	from 2 by $(\to/\underline{\to})$ again

For line 1, a more explicit annotation might run as follows: for any given formulas A, B of the object language, the metalinguistic formula at line 1 is an instance of the schema $(\underline{\mathsf{K}})$. Since 3 is established for arbitrary I, A, B we conclude that every instance of (K) is valid – though we are not formally introducing the metalinguistic universal quantifier into the discussion – and since we can evidently reason similarly in the case of the other axiom schemes, we have all we need by way of a proof of soundness for the axiom system except for showing that the rule Modus Ponens preserves validity. But we can show something stronger: not just that when formulas $A \to B$ and A are valid (true on all interpretations) so is B, but that for each interpretation I, if $A \to B$ and A are true on I then so is B. Naturally we need to help ourselves to a little bit of the homophonic semantic theory for $\vdash_{BCK}$, namely $(\to/\underline{\to})$, which can here be understood not as universally quantified but as pertaining to the specific A, B featuring in an arbitrary single application of Modus Ponens. In other words, we can show

$$\big[(\to/\underline{\to}), I \models A \to B,\ I \models A\big] \underline{\vdash}_{BCK} \big[I \models B\big] \qquad \text{(mp)}$$

For this we have worked our way from the three left-hand formula here to that on the right, using the resources of $\underline{\vdash}_{BCK}$, so, calling our starting formulas 1, 2, and 3, we may set things out vertically again, writing out the current $(\to/\underline{\to})$ explicitly for line 1:

1	$I \models A \to B \Leftrightarrow (I \models A \underline{\to} I \models B)$	
2	$I \models A \to B$	
3	$I \models A$	
4	$I \models A \underline{\to} I \models B$	From 1, 2 by (2.2c)
5	$I \models B$	From 3, 4 by Modus Ponens for $\underline{\to}$

Note that if we use the set $\{p \to q, q \to p\}$ of equivalence formulas already mentioned then the appeal to (2.2c) in line 4 would itself by an appeal to Modus Ponens for $\underline{\to}$, as in line 5.

Accordingly we conclude that whenever $A_1, \ldots, A_n \vdash_{BCK} B$, for any interpretation I, if $I \models A_1, \ldots, I \models A_n$, then $I \models B$, and not just for the case of $n = 0$: we are not just showing that the outright theorems of BCK are valid, but soundness in the stronger sense that any syntactic consequence by $\vdash_{BCK}$

of some formulas is a semantic consequence of them in the sense of being true on every interpretation on which all of them are true. And of course, there is a bit of hand-waving, already acknowledged, on the quantificational front here, but the point is to illustrate that the correspondence between aspects of the meta-logic $\vDash_{BCK}$ and the matching aspects (axioms and rules) of the object logic make the issue of soundness a trivial one. So here I am agreeing with what Brady and Rush don't say but might well have said: we have been discussing soundness, whereas what they actually say concerns a similar match-up between object and meta-logical principles in connection with a toil-free proof of *completeness* (in the general case). We return to that idea in Section 5. The reason the hand-waving over such things as the quantificational inferences required in the above discussion of soundness, as well as the inductive structure of the proof, is being taken as cause for no great concern is that by the end of the following section we will stop trying to conduct the discussion in a weakening of classical logic and look instead to an extension of classical logic for which, if Brady and Rush's remarks are correct, it should be equally trivial to provide a proof of soundness and completeness w.r.t. homophonic model-theoretic semantics. Since it is the discussion from (Brady and Rush 2009) that we are principally taking up here, a few comparative remarks on related work are postponed to the Appendix ('Bacon and others').

3 Weaken your metalogic: easier said than done

So far we have mostly gone along with the idea that it is a simple matter to shift from the use of a classical metalogic for reasoning about a weaker object logic to the use of a weaker – indeed exactly correspondingly weaker – metalogic. It is certainly easier said than done to shift to a weaker or indeed to a stronger metalogic than one finds oneself with, if "shift to" means "come to adopt as one's working logic"; some considerations highlighting the problematic nature of such a procedure can be found in (Berger 2011).[14] Here

[14]The problems seem somewhat exaggerated, in the case Kripke's envisaging a reluctance to endorse ∀E, or universal instantiation ('UI') as it is called in (Berger 2011) (see esp. p. 183 for this issue of this reluctant reasoner), on the part of someone who is not only reluctant to draw conclusions in accordance with this rule but is instead inclined to infer the negations of the instances in question ('Perverse Instantiation', p. 185). One would surely want to say of that the former reasoner does not understand what is written as "all" (or "∀") as

we are concerned not so much with endorsing a weaker logic as with being able to see what follows from what from the perspective of such a logic, when the 'what's that are involved are formulated with an indifference to details which are logically immaterial by the lights of one's customary working logic.

There are evidently many issues in the philosophy of logic involved here and the intention will be to by-pass them by changing examples to that we can examine the fate of Brady and Rush's triviality claim for completeness results w.r.t. homophonic model-theoretic semantics when the logic in question is an extension of classical logic rather than a weaker logic proposed as an alternative to classical logic. But to motivate this change of tack, something needs to be said about how in fact it is anything but a simple straightforward matter to shift – in the sense just clarified – to a weaker logic, whether or not one's starting point happens to be classical.

In fact a little has already been said, in Section 1, with the point that not every weaker-than-classical logic endorses the classical equivalence between negated universal statements and existential statements with negated content. Thus we get the oft-remarked inverse relationship between deductive power and discriminatory power.[15] In the $\neg\forall I$ vs. $\exists I\neg$ case just recalled, for instance, one could define validity$_1$ and the validity$_2$ of a formula as its truth on all interpretations for the former and as the non-existence of an interpretation which it is false in the latter. These formulations are meant to invoke the $\forall I$ and $\neg\exists I\neg$ regimentations, understood intuitionistically. It does not seem obvious to one at home in a logic in which these are equivalent which, out of validity$_1$ and validity$_2$ best represents what had always seemed to be the single undifferentiated notion of validity. In a similar vein, Kreisel, perhaps the first logician to consider the possibility of an intuitionistically acceptable completeness proof for intuitionistic (predicate) logic (in (Kreisel 1962)) by supplying a second-order notion of validity for Heyting's predicate calculus[16] and then distinguishing completeness, in respect of a formula A, as $Valid(A) \rightarrow Prov(A)$ (to use the earlier notation now for Kreisel's concepts) weak completeness in

universal quantification, while the latter, further, understands "All Fs are Gs as" as "No Fs are Gs". (Here we hedge, as in (Berger 2011), over the issue of restricted vs. unrestricted quantifiers.) Cases of disagreement in reasoning according to rules which go beyond those uniquely characterizing the logical constant in question would, however, be more problematic. (See subsection 4.32 of (Humberstone 2011).) The same applies to Lewis Carroll's prototype, of the tortoise reluctant to reason in accordance with Modus Ponens; in this case a loose parallel to Kripke's Perverse Instantiation is provided by Haack's 'Modus Morons' (in (Haack 1976)).

[15]For references, discussion, and some caveats, see (Humberstone 2007a).

[16](Dummett 1977, §5.6) provides further discussion of this and other approaches.

respect of A as $\sim\sim(\mathit{Valid}(A) \rightarrow \mathit{Prov}(A))$, lifting this terminology to a logic (in particular to Heyting's) when the corresponding property is possessed in respect of all formulas. (Note incidentally that this has nothing to do with the usual 'derivability' vs. 'outright provability' contrast between strong and weak completeness or between the similarly named properties from (Bacon 2013), touched on in the longer notes on Section 2 in the Appendix, headed 'Bacon and others'.)

What the adherent of a weaker logic regards as the apparatus for making finer discriminations in the subject matter than are available to the adherent of the stronger logic, who is faced instead with a proliferation problem: how on earth to respond the requirement to 'deconflate'. This of course arises not just with characteristically metalogical vocabulary such as *valid* and *complete* (as it does in other specific areas: intuitionistically we have non-empty vs. inhabited sets/species, apartness vs. non-identity of real numbers and so on), but to the core logical vocabulary itself. Sticking with the case of intuitionistic logic as seen from a classical perspective, should one represent "Only Fs are Gs" as "$\forall x(Gx \rightarrow Fx)$" or as "$\forall x(\neg Fx \rightarrow \neg Gx)$"? Or again "$\neg\exists x(Gx\wedge\neg Fx)$? Even though as candidate representations from the point of view of English syntax, the last two might appeal on the grounds that the negations involved may explain the appearance in only contexts of negative polarity items – "She only ever visits on Fridays" – semantically and logically one is not accustomed to making much of a fuss about which to write. In fact the first pair here raise another specifically metalogical choice for how to construe completeness for the case in which a contraposed conditional is not equivalent to the original option remaining after a notion of validity has been settled on: do we require that for all formulas A, $\mathit{Valid}(A) \rightarrow \mathit{Prov}(A)$, or instead that for all A, $\neg \mathit{Prov}(A) \rightarrow \neg \mathit{Valid}(A)$ (with a corresponding distinction for soundness). Although what one usually wants from anything that would be pronounced as "if and only" if would be a connective with the equivalential properties described in Section 2, favouring the choice of $A \leftrightarrow B$ as standardly defined by $(A \rightarrow B) \wedge (B \rightarrow A)$ rather than the intuitionistically non-equivalent $(A \rightarrow B) \wedge (\neg A \rightarrow \neg B)$, which does not support replacement in the same way, who is to say for which occurrences in one's own work a material biconditional should give way to the latter biconditional instead – not to mention other classically equivalent but intuitionistically non-equivalent candidates – $(A\wedge B)\vee(\neg A\wedge\neg B)$, and so on? A further example of this phenomenon is given in the Appendix (see 'Bell and others').

An *embarras de richesses* arises at another level also, when, given a theory with classical logic as its background logic, an attempt is made to specify a corresponding theory with a weaker logic in the background. Thus, for example, (Kreisel 1958), p. 371*f.* writes "The theory is an extension of Heyting's arithmetic, by which we mean the system obtained by adding the non-logical axioms of classical arithmetic to Heyting's predicate calculus." Here we are not making the familiar point that citing a theory does not suffice to single out any particular axiomatization of it. What is currently problematic, rather, arises from citing a particular axiomatization – and in the context of Kreisel's discussion, the Dedekind–Peano axioms are no doubt what is intended – and then saying that one uses 'these very axioms' with a weaker background logic (here intuitionistic predicate logic) to axiomatize the corresponding weaker theory. And the problem is that although we are talking about the specifically non-logical axioms, they are shot through with logical vocabulary, and different theorists may have made different choices as their official account of the latter vocabulary, differences which are completely immaterial with classical logic in the background, for example regarding $A \to B$ as abbreviating $\neg A \vee B$, or as abbreviating $\neg(A \wedge \neg B)$, or as primitive – in the last case perhaps alongside $\neg$ with $\wedge$ and $\vee$ takes as defined in any convenient way – but which of these is taken over into the (for the present example) intuitionistic case makes a great difference in view of their non-equivalence according to the weaker logic. That is, there is really no such thing, from the perspective of the weaker logic, as *the* non-logical axioms to be used for the weaker theory. To illustrate the point with a weakening of classical logic closer to Brady's heart (though far too slight a weakening for his own tastes), we might recall the issues arising (see (Anderson, Belnap and Dunn 1992, 430)) over whether to formulate an induction schema in relevant arithmetic (with underlying logic **R**) as

$$\big(A[0] \wedge \forall x(A[x] \to A[x'])\big) \to \forall x A[x],$$

where "x'" is for the successor of x, on the one hand or instead as

$$A[0] \to \big(\forall x(A[x] \to A[x']) \to \forall x A[x]\big),$$

on the other. A related contrast (conjunctive antecedent vs. iterated implication) has been discussed in the literature on the identity relation in relevant logic: which is the appropriate way of expressing the transition from an identity statement $t = u$ along with a statement involving t to the corresponding conclusion with u? (See (Mares 1992), (Kremer 1999).)

To avoid all such challenges, let us switch to a setting in which as much of classical logic is available in the metatheory. Indeed, one of the author's students (Joel Towell) asked the following question on first reading Section 5 of (Brady and Rush 2009): What would they say about what they present as trivial 'toil-less' semantics if it were presented in this style for *classical* propositional logic itself? A good question: indeed, what difference is there between perfectly orthodox presentation of the semantics – except perhaps that interpretations might be called (Boolean) valuations – and the much-maligned trivial semantics? Soundness and completeness of a particular proof system w.r.t. this semantics does not seem trivial in the sense of being completely uninformative. But the classical switch I have in mind is not quite this because the aim is not so much to rebut the suggestion that homophonic model theoretic semantics is pointlessly uninformative as the suggestion that it is automatically available, with soundness and completeness results effortlessly forthcoming. For this purpose, we need an extension of classical propositional logic, which in view of its Post completeness, will have to be couched in an expansion of its language. The simplest such expansion is the language of monomodal logic, so let us go there now, and show the trouble created for 'semantics without toil' by a phenomenon that is particularly well known in that arena: Halldén incompleteness. (Despite the name, and like Post completeness and incompleteness, Halldén completeness and incompleteness are syntactical notions, their semantic repercussions notwithstanding – though the repercussions envisaged are very different for Kripke semantics than they are for the kind of semantic interpretation Halldén has in mind: see Section 6.[17])

[17](Prior 1957, 54), referring to his proposed modal logic Q, writes "Dr. Alan Ross Anderson has pointed out to me, however, that Q can be shown to be 'semantically complete' in the sense of S. Halldén, i.e., it contains no theses of the form $A\alpha\beta$ in which α and β have no variables in common and yet neither is a thesis on its own." (Here Prior is writing $A\alpha\beta$ for $\alpha \vee \beta$.) This "i.e." is quite misleading. The syntactic property alluded to here is not what Halldén calls semantic completeness, the latter signifying for him (as usual) a link between provability and (a suitable notion of) validity, which Halldén argues cannot be forged when there are provable variable-disjoint disjunctions neither disjunct of which is provable. We will look at this argument in its original setting in the Postscript below (Section 6), with Section 4 concentrating on its bearing on semantic completeness *à la* Brady and Rush.

4 Halldén completeness and semantic completeness

Proceeding with the project just outlined, let us begin by recalling the most widespread contemporary understanding of Halldén completeness for propositional logic. It applies to logics as sets of formulas – historically, originally to modal logics (Halldén 1951) – and if we are thinking of logics as consequence relations, its application depends only on the consequences of the empty set; it presumes the presence of a disjunction connective with more or less the expected properties. In terms of consequence relations those properties would amount to a disjunction $A \vee B$ having for is consequences those formulas which are both consequences of A and consequences of B.[18] However, as just remarked, its application to a consequence relation depends only on the consequences of $\varnothing$, so we are in effect thinking of logics as sets of formulas and the elements of such set will be described as provable in the logic.

Definition 4.1 A logic is *Halldén complete* iff whenever $A \vee B$ is provable, and there are no sentence letters appearing in both A and B, then either A or B is provable.

Remark 4.2 Halldén's original definition (in (Halldén 1951)) was a little different, having the additional requirement that each of A, B was constructed using a single propositional variable, but this is no longer regarded as a useful demand to build into the definition, though sometimes it is built in with the aid of a qualification: *strongly* Halldén complete. (See (Schumm 1993a), appearing also as the appendix to (Schumm 1993b), and references there for further discussion.) ◀

Halldén's idea was that for a plausible notion of validity to coincide with provability in a logic, the logic would have to be Halldén complete, and though his argumentation in (Halldén 1951) seems somewhat mysterious today: further detail on this front is postponed to Section 6, for after we have seen the bearing of these issues on Brady and Rush's discussion. Halldén completeness has been discussed in connection with classical propositional logic, numerous

[18] The usual understanding of $\vdash$'s "having disjunction" would be stronger: that $\Gamma, A \vee B \vdash C$ iff $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$, but for present purposes we do not need to require this for $\Gamma \neq \varnothing$.

intermediate logics – though not for those which, like intuitionistic propositional logic itself, have the disjunction property (understood as per Definition 4.1 except without the condition that A, B have no common sentence letters – making Halldén completeness an immediate consequence of this property), modal logics,[19] relevant logics (Routley and Meyer 1972, 71), and modal relevant logics (Mares 2003), among others. Here our focus is on (classically based) modal logics, as Halldén's was in (Halldén 1951). The parenthetical "classically based" here reflects the fact that the metalanguage to be used for reasoning about such propositional logics in homophonic model-theoretic semantics will freely avail itself of the deductive resources of classical predicate logic, side-stepping the difficulties noted in Section 3. And in fact this reasoning will be rather familiar[20] from the case of showing that the set of classical tautologies, is Halldén complete: if neither A nor B is tautologous, so there are Boolean valuations v, v' assigning the value F to A, B respectively, then if A, B have no sentence letter in common we can 'combine' v and v' into a single Boolean valuation v'' matching v's assignments of T, F to the sentence letters in A and matching v''s assignments to those in B, and we will then have $v''(A \vee B) = F$, revealing this disjunction not to be tautologous.

For the case of interest to us here we have interpretations in place of valuations and we have an additional 1-ary connective $\Box$ to contend with; this notation is reasonable in view of the traditional very generous definition of what a (mono)modal logic is, though here we extend the idea from logics as sets of formulas to logics as consequence relations.[21] Concerning $\Box$, our semantic account tells us (as in Example 2.1), that for all interpretations I and all formulas A:

[19] (Chagrov and Zakharyaschev 1991, 201–206) and (Chagrov and Zakharyaschev 1993, 990–997) provide a wealth of information on Halldén completeness among intermediate as well as modal logics. One historical correction: on p. 201 of (Chagrov and Zakharyaschev 1991) we read that the existence of Halldén incomplete intermediate logics was pointed out in (Halldén 1951). The only intermediate logic in the sense in which Chagrov and Zakharyaschev use the phrase – i.e. for (consistent) superintuitionistic logics – mentioned by Halldén is intuitionistic propositional logic itself, concerning which he makes the point about the disjunction property made in the text here. The authors have perhaps been influenced by the wording of Theorem 2 in (Halldén 1951), with its reference to "S1, S3, or any intermediate calculus," in which it is clear from the context that the last three words mean any logic ('calculus') between S1 and S3.

[20] Given, to cite only discussions already in our bibliography, on the opening page of (Mares 2003), and on p. 862 of (Humberstone 2011).

[21] In this setting we can think of $\vdash$ as a modal logic if $\vdash$ is a substitution-invariant extension in an expansion (by the addition of $\Box$) of the consequence relation of classical propositional logic, formulated with some functionally complete stock of connectives.

$$I \models \Box A \Leftrightarrow \underline{\Box}(I \models A) \qquad (\Box/\underline{\Box})$$

Here we see "⇔" putting in its regular appearance, so to make life simple we may suppose that the object logic we are currently treating is some Halldén incomplete (normal) extension of $\mathsf{S4}$, in which case we can take ⇔ as strict equivalence.[22] For definiteness let us pick the logic mentioned in connection with Schumm in note 22 and give this logic the ad hoc label S for present purposes, for which purposes nothing about its identity natters beyond its being a normal Halldén incomplete normal extension of $\mathsf{S4}$.

Lemma 4.3 *If $p_1, \ldots, p_k$ are the only sentence letters appearing in the formula A and I, I' are interpretations for which we have $I \models p_i$ iff $I' \models p_i$ $(i = 1, \ldots, k)$ then $I \models A$ iff $I' \models A$.*

What follows is a relatively informal proof of Lemma 4.3, though in due course we will write ⇔ in place of "iff" so as to recall the conditions this connective is presumed to satisfy, according to the current consequence relation $\vdash_{\mathsf{S}}$, taken as the local consequence relation associated with our sample logic S.[23] Its metalogical counterpart $\underline{\vdash}_{\mathsf{S}}$ will have to be a corresponding modal predicate logic, since we need the resources of quantification, and for simplicity let

[22] Normal modal logics (as sets of formulas) are sets of formulas in a language with 1-ary $\Box$ and some functionally complete set of Boolean connectives, closed under Uniform Substitution, necessitation (prefixing of $\Box$) and Modus Ponens, and containing all truth-functional tautologies and the formula $\Box(p \to q) \to (\Box p \to \Box q)$; modal logics in general are similarly defined but without the condition concerning this last formula or the condition pertaining to necessitation. The normal extension of a modal logic is the least normal modal logic extending that logic. The supposition that we are dealing with a Halldén incomplete normal extension of $\mathsf{S4}$ presumes that there are such logics, so for definiteness, we may take an example suggested by (Lemmon 1966, 297), namely the normal extension of $\mathsf{S4}$ by: $(p \to \Box\Diamond p) \vee \Diamond(\Box q \vee \Box\neg q)$, in which "$\Diamond$" abbreviates "$\neg\Box\neg$". Lemmon's actual suggestion does not include the word 'normal', for which reason a more straightforward example is the extension of $\mathsf{S4.4}$ – of necessity normal (Segerberg 1975) – by the closely related axiom $\Box(\Diamond p \to \Box\Diamond p) \vee \Box(\Box\Diamond q \to \Diamond\Box q)$, given in (Schumm 1969) in response to the question of whether there were any modal logics strictly between $\mathsf{S4}$ and $\mathsf{S5}$. (For further information, see Humberstone 2016, 417f.) (Schumm 1993b, 202) notes the existence of uncountably many Halldén incomplete extensions of $\mathsf{S4}$, and following up the reference he gives there shows that we may insert "normal" before "extensions" here. As Lemmon also mentions, J. C. C. McKinsey had earlier observed that there are no Halldén incomplete extensions of $\mathsf{S5}$, and indeed had already isolated (McKinsey 1953, 113) a Halldén incomplete logic between $\mathsf{S4}$ and $\mathsf{S5}$. Further pertinent material can be found in (Schumm 1975).

[23] The local consequence relation associated with a modal logic when the latter is construed as a set of formulas is the relation holding between Γ and A when A can be obtained from theorems of the logic and formulas in Γ by (successive) applications of Modus Ponens. The label is appropriate because semantically this is a matter of truth at a point in a model, by contrast with global consequence, requiring instead only preservation of the property of being true throughout a model, a syntactic gloss on which would have to allow applications not only of Modus Ponens but also of Necessitation.

us suppose that $\Box$ commutes with $\forall$ according to this consequence relation (the Barcan equivalences). Evidently the basis case for a proof by induction on the complexity (= number of primitive connectives used in the construction of) A is given by the condition on I, I', since in this case A is one of the p_i. So suppose that A is $\#(A_1, \ldots, A_n)$ for some n-ary connective $\#$. In fact in the present case we can take $n \leq 2$ since we may suppose that our object language connectives are just, for example $\neg$, $\rightarrow$ and $\Box$, and for illustrative purposes let us look at the $\Box$ case, since this is the only novelty not present in the "combine the Boolean valuations" argument recalled above. Thus our inductive hypothesis is that for a formula A and interpretation I:

$$I \models A \Leftrightarrow I' \models A \tag{4.1}$$

and we want to conclude that

$$I \models \Box A \Leftrightarrow I' \models \Box A \tag{4.2}$$

But by (2.2d), (4.1) has as a $\underline{\vdash}$-consequence $\underline{\Box}[I \models A] \Leftrightarrow \underline{\Box}[I' \models A]$ from which we get (4.2) via ($\Box/\underline{\Box}$) (and the replacement properties of $\Leftrightarrow$).

Remark 4.4 As the above informal argument shows, the wording of Lemma 4.3 is potentially misleading, especially in view of the usual expectations one might have on seeing the abbreviation "iff" – in particular as suggesting that we have here a material biconditional. The point is rather that a suitably generalized version of (4.2) is a $\underline{\vdash}$-consequence, by our metalogical $\underline{\vdash}_{\mathsf{S4}}$-extending consequence relation $\underline{\vdash}$ (which in the case of interest below will be $\underline{\vdash}_{\mathsf{hi}}$), of (4.1). It would similarly be potentially confusing to read Lemma 4.3 as saying that whether an interpretation I verifies a formula A is determined by (or depends at most on) whether or not it verifies p_i for the various p_i occurring in A. That might suggest that if A is constructed from one sentence letter p alone, say (for simplicity) p, and so as to have a special case of (4.1)–(4.2) before us, that A is $\Box p$ and $I \models p$ and as it happens, also $I \models \Box A$. Now suppose we have another interpretation I' for which, as with I we also have $I' \models p$. Are we supposing it follows (in the envisaged metalogic) that since I and I' thus 'agree' on p, each verifying it, that I and I' must similarly agree on $\Box p$, and thus, since $I \models \Box p$, we must have $I' \models \Box p$? That would be disastrous since it would make the context $\Box(\cdot)$ – and indeed all contexts – left-extensional in the sense of satisfying the condition (here stated for $\Box(\cdot)$)

for all formulas B, C and $\vdash$ sound and complete w.r.t the current semantics:

$$B, C, \Box B \vdash \Box C \tag{LE}$$

the terminology here being taken from (Humberstone 2011, 454ff.); the same goes for the 'right extensionality' principle (RE) in play in that discussion. This would follow if we had $B, C \vdash B \Leftrightarrow C$, but in general we do not. (Recall we are taking $\vdash$ as the local consequence relation of an associated with on extension of $\mathsf{S4}$ and $\Leftrightarrow$ as strict equivalence. It is because the latter is already in the object language that we have used it in the hypothetical schema $B, C \vdash B \Leftrightarrow C$. What really matters is that we do not have this principle in the metalogic, which would force its validity in the object logic via $(\Box/\underline{\Box})$.) This in the relevant sense of 'agreement' – the sense in which agreement on components yields agreement on compounds – that fact that I and I' both verify a formula A both fail to verify A it does not suffice for them to agree on A: for this we require $I \models A \ \Leftrightarrow \ I' \models B$. Another way – aside form misconstruing the "iff", that is – of falling into this trap would be to adopt the notation warned against in Remark 1.1, and writing "$I(A) = T$" in place of "$I \models A$". Then the initially given details of the case above, namely that $I \models p$, $I' \models p$, and $I \models \Box A$, become $I(p) = T$, $I'(p) = T$ and thus that $I(p) = I'(p)$, which makes it looks as though I and I' cannot but treat p alike, even as it occurs embedded with the context $\Box(\cdot)$, and hence that we would then be committed to $I(\Box p) = I'(\Box p)$. To resist such temptations we have Frege's distinction of sense and reference to help, though this is not perhaps a remedy that readily comes to mind in model-theoretic semantics in which possible worlds are not being invoked. (When they are, we have for Frege's distinction, the Carnapian substitute of the intension/extension distinction, but the latter assigns a semantic value as the extension of an expression only relative to a world in a model or interpretation, whereas here we are saying $I(p) = T$ with no such relativization.) ◀

Since there are no constraints on how interpretations treat sentence letters (aside from the ill-fated (*) from Section 1), which in the present setting would not count as a constraint anyway, we can re-run the 'combination of valuations' argument for Halldén completeness in the non-modal case, reprised above. Since we are just concerned here what is provable outright according to $\vdash_{\mathsf{S}}$, we refer to the logic as S:

Proposition 4.5 *The logic* S *is not sound and complete w.r.t. its homophonic model-theoretic semantics.*

Proof. Suppose that S is sound and complete w.r.t. its homophonic model-theoretic semantics. Take a disjunction $A \vee B$ whose disjuncts share no sentence letters and for which (i) $\vdash_{\mathsf{S}} A \vee B$, (ii) $\nvdash_{\mathsf{S}} A$, and (iii) $\nvdash_{\mathsf{S}} B$ (for example, the $A \vee B$ from Schumm mentioned in note 22). In view of (ii) and (iii) and the supposed completeness of the logic, there are interpretations I, I' with $\underline{\neg} I \models A$ and $\underline{\neg} I' \models B$. Since A and B have no sentence letters in common, we may consider an interpretation I'' which treats the sentence letters in A as I does, and those in B as I' does, in which case $\underline{\neg} I'' \models A$ and $\underline{\neg} I'' \models B$, and then by Lemma 4.3, $\underline{\neg}(I \models A \mathrel{\underline{\vee}} I \models B)$, so, finally, $\underline{\neg} I \models A \vee B$, thus contradicting the supposed soundness of the logic, in view of (i). □

Fixing on a particular modal logic as $\vdash_{\mathsf{S}}$ was of course just done for definiteness: this argument works for any normal Halldén incomplete extension of S4, and since soundness is not problematic (as Section 2 aimed to show), what is really revealed is a tension between the syntactic property of Halldén incompleteness and the semantic notion of completeness provided by homophonic model theory. One might take, for whatever reason, a dim view, as Halldén clearly did, of Halldén incomplete logics, but that is not really to the point here, which was to query the automatic availability claimed for this semantic approach in the initial quotation from (Brady and Rush 2009).[24] In Section 6 we look more closely at Halldén's original grounds for dissatisfaction with the logics – especially C. I. Lewis's S1, S2, S3 – displaying this syntactic property: they are strikingly similar for the grounds provided in the proof of Proposition 4.5 for concluding that S was not amenable to a homophonic semantic treatment, though Halldén's point was couched in different terminology ('normal interpretations': see Section 6), and he may have overestimated Lewis's commitment to such an interpretation in claiming that these weaker logics were semantically intractable *by Lewis's own lights.*

For the remainder of this section let us attend to a qualm that might be raised about excessive hand-waving in the proof of Proposition 4.5: where precisely do we get the metalogical resources for the crucial ∀∀∃ step? This

[24] A referee (or expert reader) points out – something I had completely forgotten – that the present considerations are closely connected with those urged in §3.3 of (Williamson 2013) to urge that the set of what are there called *metaphysically universal* formulas should constitute a Halldén complete modal logic.

claims the existence of a suitable I'' on the basis of suitably related I, I', whereas we have not spelled out explicitly any principles for a formalized metatheory which would secure such existential claims. Rather than doing ourselves to that, however, we shall see, instead, how to sidestep the apparent need to do so by giving a version of the argument which, though it does not apply to the S under consideration above, readily finds application among the normal (as well as non-normal) modal logics. To move in that direction, we begin by noting a close relative of Halldén incompleteness which has us temporarily jettisoning our recent classical underpinnings.

This close relative, not usually thought of in the same breath as Halldén completeness and incompleteness, is the property of consequence relations called *uniformity* in (Łoś and Suszko 1958). Not usually – but occasionally: the connection is made explicit by Shoesmith and Smiley in the works cited below, where a uniformity-like condition appears under the name of 'cancellation' which for finitary consequence relations can be taken in the form, with $vbl(\cdot)$ maps a set of formulas to the set of sentence letters occurring in those formulas :

If $\Gamma, \Delta \vdash A$ and $vbl(\Gamma \cup \{A\}) \cap vbl(\Delta) = \varnothing$, then $\Gamma \vdash A$ or else $\Delta \vdash C$ for all C.

While this condition needs considerable elaboration to cover the case of arbitrary (i.e., not necessarily finitary) consequence relations, it suffices as it stands for present purposes as a necessary condition for a consequence relation to be many-valued in the sense of being determined by a single (logical) matrix. (The elaborations are given in Shoesmith and Smiley 1971 and had already been noted by Ryszard Wójcicki as a necessary supplementation to the treatment in Łoś and Suszko 1958: for the rather complicated details here, see Wójcicki 1974, and p. 278 of Shoesmith and Smiley 1978.)

To see the connection between the claim that a matrix-determined $\vdash$ must satisfy the uniformity/cancellation inset above and the argument that classical propositional logic is Halldén complete, look at what happens when the condition is not satisfied for such a $\vdash$: the matrix evaluations that end up witnessing the fact that $\Gamma \nvdash A$ and $\Delta \nvdash C$, for some C, assign (i) designated values to the formulas in Γ but an undesignated value to A and (ii) designated values to the formula in Δ, so since Δ is variable disjoint from $\Gamma \cup \{A\}$ these assignments can be combined into a single evaluation showing that $\Gamma, \Delta \nvdash A$. (Shoesmith and Smiley 1978, 618f.) go on to explain the relationship to Halldén completeness, as standardly understood. (There is no mention of Halldén completeness in the authors' Shoesmith and Smiley 1978.) Kracht (1999, 29) boldly redefines Halldén completeness as a property of consequence relations not making any reference to disjunction – though he does mention the usual definition later (p. 141) – but instead using a special case of the inset condition above with $\Gamma = \varnothing$ and $\Delta = \{B\}$, and accordingly announcing (Theorem 1.6.5, p. 30) that all matrix-determined consequence relations are Halldén complete, without the expected further condition that the operation associated with disjunction in the algebra of the matrix in question take any pair of undesignated values to an undesignated value.[25]

Whatever one calls it, this cancellation/uniformity property is enforced, no less than Halldén completeness *stricto sensu* is, by the homophonic model-theoretic semantics, via the same "combine the distinctive features of I and I' within a single interpretation I''" reasoning (even though, yes, we are heading towards a case in which we can by-pass this reasoning). The property in question is famously not possessed by (Johansson style) minimal logic, for whose consequence relation $\vdash_{\mathsf{ML}}$ we have for example $p, \neg p \vdash_{\mathsf{ML}} \neg q$, even though $p, \neg p \nvdash_{\mathsf{ML}} r$ (and thus we do not have $p, \neg p \vdash_{\mathsf{ML}} C$ for all C) and $\nvdash_{\mathsf{ML}} \neg q$. The violation is even more dramatic if we take a formulation with $\bot$ as primitive (in terms of which $\neg A$ would be defined as $A \to \bot$), but about which $\vdash_{\mathsf{ML}}$ says nothing it does not say about all formulas. Then there are no shared sentence letters on the left and the right in the claim that $\bot \vdash_{\mathsf{ML}} \bot$ since there are no sentence letters at all, yet we have neither $\vdash_{\mathsf{ML}} \bot$ nor: for all C, $\bot \vdash_{\mathsf{ML}} C$. In this case, much discussed in (Shoesmith and Smiley 1978), no determining matrix is available because making the value corresponding to the nullary connective $\bot$ designated would conflict with the former and making it undesignated would conflict with the latter. The appropriate incarnation of the general ($\#/\underline{\#}$) form for the case of # as $\bot$ in homophonic model theory

[25] This condition was part of the definition of a *normal* matrix used in (Kripke 1965), in which one reads, concerning the property of Halldén incompleteness, that "Halldén essentially showed that no system with his property can possess a normal characteristic matrix." (Kripke refers the reader to Church's definition of normal matrices or, as Church calls them – following Carnap – normal interpretations: (Church 1956, 117); Halldén has his own definition of 'normal interpretation', quoted in Section 6 below. Normality of matrices in this sense has nothing to do with normality of modal logics, whether in Kripke's originally defined sense – requiring the provability of all formulas of the form $\Box A \to A$ – or in the subsequently adopted sense explained in note 22, which omitted this requirement. The former pertains to the behaviour of the non-modal vocabulary and the latter – essentially – to that of the modal vocabulary.) Note that the algebras used to define the logics on which (Bacon 2013) focuses do not satisfy the condition: the top (and sole designated) element 1 is typically a lattice join of elements distinct from 1.

would then be:

$$[\, I \models \bot \,] \;\;\Leftrightarrow\;\; \underline{\bot} \qquad (\bot/\underline{\bot})$$

in which $\Leftrightarrow$ can be taken as $\underline{\leftrightarrow}$, recalling that the $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$-fragment of minimal logic coincides with that of intuitionistic logic. At this point we observe that "I" does not appear free on the right of $(\bot/\underline{\bot})$, so we have the consequence that for all interpretations I, I':

$$[\, I \models \bot \,] \;\;\Leftrightarrow\;\; [\, I' \models \bot \,] \qquad (4.3)$$

from which we might be tempted to conclude that either all interpretations verify $\bot$, in which case we should have $\vdash_{\mathsf{ML}} \bot$, or none do, in which case we should have $\bot \vdash_{\mathsf{ML}} C$ for all C, so since neither of these alternatives obtains, minimal logic presents another example in which we do not have soundness and completeness w.r.t. the homophonic model-theoretic semantics. But this would be to forget that the metalogical reasoning is supposed to be as licensed by, for the present case, $\underline{\vdash}_{\mathsf{ML}}$, and this inference from (4.3) to a disjunction of universal claims is not thus licensed. This was exactly the kind of nuisance stressed in the previous section occasioned by working not only on but in a logic weaker than one is accustomed to, prompting us to consider classically based ventures.

Undeterred, however, we proceed as with Halldén completeness proper, and shift to a classically based modal logic, so that in the metalogic all of classical predicate logic is available for exploitation.[26] More specifically, let us pass to the smallest normal modal consequence relation $\vdash_{\mathsf{K}}$ (the local consequence relation, in the sense of note 23, associated with the smallest normal modal logic, K[27]), since in this logic the 'pure' ('letterless') formula $\Box\bot$ enjoys the same status – neither provable nor refutable – as $\bot$ itself does in minimal logic.[28] There is a slight complication in that there is no endogenous $\Leftrightarrow$, so $\underline{\vdash}_{\mathsf{K}}$ will have to take this on as a new connective, $p \Leftrightarrow q$ being given the powers it would have had it been defined infinitarily as the conjunction of all

[26]Considered as a consequence relation this means that require not only that we have an extension (in a monomodal language) of the consequence relation of classical predicate logic but that it should satisfy also the usual *conditional* classicality conditions on consequence relations – such as $\rightarrow$-classicality and $\vee$-classicality as they are called in (Humberstone 2011).

[27]"K", by tradition, for $\underline{\underline{\text{K}}}$ripke here – no connection with the $K/(\mathsf{K})$ of *BCK*.

[28]In the terminology of (Kracht 1999, 88), $\Box\bot$ is a non-trivial constant (of K); see also the top of p. 142 there, down to and including Proposition 3.7.7 (something mention by name as the online version of the book is differently paginated).

the $\Box^n(p \leftrightarrow q)$ $(n \geq 0)$. Invoking both $(\underline{\bot}/\bot)$ and $(\underline{\Box}/\Box)$, we get

$$[\, I \models \Box\bot \,] \;\Leftrightarrow\; \underline{\Box}\,\underline{\bot} \tag{4.4}$$

Corresponding to (4.3) above, and for the same reasons, we have (for all I, I'):

$$[\, I \models \Box\bot \,] \;\Leftrightarrow\; [\, I' \models \Box\bot \,] \tag{4.5}$$

and hence, first weakening the $\Leftrightarrow$ in (4.5) to $\underline{\leftrightarrow}$, this gives us a premiss from which we may conclude:

$$\forall I(I \models \Box\bot) \,\underline{\vee}\, \forall I(\underline{\neg} I \models \Box\bot), \tag{4.6}$$

since our reasoning is now entirely classical. For the record then, we have, written in the same style as Prop. 4.5:

Proposition 4.6 *The logic* K *is not sound and complete w.r.t. its homophonic model-theoretic semantics.*

Proof. According to (4.6), one or other disjunct of the object language disjunction $\Box\bot \vee \neg\Box\bot$ is valid, but since neither is provable in K, we have a failure of semantic completeness. $\Box$

The main point of interest here, as foreshadowed earlier, is that we did not need – as we did for the proof pf Proposition 4.5 – the step that combined one I with a second I', occasioned by the invalidity of the disjuncts, into a single I'', because the r.h.s. of (4.4) does not contain "I", making it what it says about its l.h.s. independent of any particular interpretation (4.5). Note that in moving from minimal logic into classically based modal logic to illustrate problems coming from the uniformity/cancellation effects of homophonic model theory we have got enough classicality of board for Kracht's redefined version of Halldén completeness, a special case of uniformity/cancellation, to coincide with Halldén completeness. In the present instance we have the 'Halldén unreasonable' disjunction $\Box\bot \vee \neg\Box\bot$ K-provable without either disjunct's being provable, despite their not sharing a sentence letter.[29]

Digression. Although our concern here is with propositional logics (as object logics) since we are following the lead of Brady and Rush, and it is not

[29] If you want to see some sentence letters instead of "$\bot$" and make this same point, then take distinct letters, p and q, and replace the first "$\bot$" with "$p \wedge \neg p$" and the second with "$q \wedge \neg q$".

quite clear what they would have in mind as a similarly toil-free treatment in the case of predicate logics, it is perhaps worth recalling that (for convenience again) classical predicate logic with identity provides us with sentences constructed (like $\Box\bot$ in the example just considered) without the aid of non-logical vocabulary, such as $\exists x \exists y (x \neq y)$ – call this "$\not\models$" – which the logic does not decide (does not prove or refute) and thus with $\not\models \vee \neg\not\models$ we have a Halldén unreasonable disjunction one of whose disjuncts must be true on all interpretations despite its unprovability – at least if the treatment of predicate logic is close to that of propositional logic. To increase the closeness, we could remove "=" from the language, and even the quantifiers as well, and take $\not\models$ as primitive, as the sole frill added to zero-order logic, this sentential constant being interpreted in conventional model-theoretic terms as true in structures with a universe of two or more elements. (Suitable conditions on $\vdash$ are then given by the unconditional requirement (i): $A(t), \neg A(u) \vdash \not\models$, where $A(t)$ and $A(u)$ are any formulas differing in that one has t at one or more positions in which the other has u, and the conditional requirement (ii): if $\Gamma, Ft, \neg Fu \vdash C$, then $\Gamma, \not\models\ \vdash C$, where the monadic predicate letter F and the terms t, u do not occur in the formulas of $\Gamma \cup \{C\}$.) ***End of Digression.***

Returning to propositional logics themselves, let us note that Proposition 4.6 is just one application of something more general. Obviously we get a similar result for K4, K5, and in general for all the normal modal logics that do not extend KD or KVer, where the latter is the least (normal) modal logic containing $\Box\bot$, since among normal modal logics a logic decides all pure formulas (i.e., proves or refutes all formulas constructed without sentence letters) if and only if it proves $\Diamond\top$ or its negation. But this consideration, and a proof as for Proposition 4.6 above, extend outside of the sphere of normal modal logics, as we may record here:

Proposition 4.7 *No modal logic which fails to decide all pure formulas is sound and complete w.r.t. its homophonic model-theoretic semantics.*

Proof. Suppose we have a modal logic which does not decide the pure formula A. Then reasoning as in the proof of Prop. 4.6 but with (4.7) in place of (4.6):

$$\forall I(I \models A) \veebar \forall I(\underline{\neg} I \models A), \tag{4.7}$$

we have one or other of $\neg A$ valid but *ex hypothesi* not provable in the logic in question. □

Remark 4.8 As it bears on non-normal modal logics which are not extensions of the smallest congruential modal logic, Proposition 4.7 is sensitive to the choice of the underlying non-modal logic, since if there are no nullary connectives ($\top, \bot$) in that language there are no pure formulas available and the equivalences exploited in note 29 is not available to simulate such formulas. As (Makinson 1973) points out, the smallest modal logic in the language with no pure formulas is Halldén complete, while the smallest such logic in the language providing such formulas is Halldén incomplete. (This gloss on (Makinson 1973) makes use of the characterization given in Proposition 6.7(*iii*), and of the 'congruential' terminology from (Makinson 1971) to apply to modal logics – as sets of formulas – containing $\Box A \leftrightarrow \Box B$ whenever they contain $A \leftrightarrow B$.) ◀

We mention one interesting application of Proposition 4.7 among the non-normal modal logics for which the issue raised in Remark 4.8 does not arise:

Example 4.9 (Irvine 2013) extols the virtues of S7 with part of the informal motivation coming from a distaste for Halldén incompleteness, the Abstract heading the paper beginning:

> Following Halldén, we define S7 as the system generated by the addition of $\Diamond\Diamond p$ to S3. Initial motivation for the extension comes from Halldén's paradox. In addition to resolving the paradox, the resulting system generates a helpful framework for comparing classical propositional logic [with various alternative (non-modal) logics].

Similarly on p. 525 we read:

> Initial motivation for systems such as S7 thus lay in the fact that S1–S3 have a certain kind of incompleteness, namely that they contain theorems of the form $\alpha \vee \beta$, such that α and β have no propositional variable in common and yet neither α nor β is a theorem.

And on the following page, noting that one of the disjuncts of some familiar disjunctions used to illustrate Halldén incompleteness extends S3 to S4 and the other to S7 (cf. Proposition 6.7(*iii*) in our Postscript), Irvine writes:

> The resulting systems are, respectively, S4 and S7. In either case Halldén's paradox will be resolved.

The talk of paradox in this connection derives from the discussion in (Hughes and Cresswell 1968, 268ff.) where the existence of Halldén incomplete modal logics is described as something that "would initially seem strange" and then as seeming "paradoxical at the very least" (both on p. 268), and then (p. 269) they describe Halldén complete logics as those for which the paradox will not arise. It is clear from Hughes and Cresswell's overall discussion, however, they do not wish to commit themselves to finding Halldén incompleteness paradoxical, but rather as reflecting neutrality as between two ways of settling a question: which disjunct to endorse (of a disjunction witnessing Halldén incompleteness). See again the Postscript to the present paper. Irvine appears to be simply assuming that because for the Halldén incomplete S3 we have S3 = S4 ∩ S7, S4 and S7 are themselves bound to be Halldén complete. While this was long known to be correct for S4 (Theorem 2 in McKinsey 1953), it was observed not to be the case for S7 in §4 of (Shukla 1972) (and, in the same year, in (Pledger 1972), final paragraph), which rather undermines at least this aspect of Irvine's case for S7. Shukla's observation can be put in terms of Proposition 4.7 above by noting that S7 does not decide the pure formula $\Box\Diamond\Diamond\bot$ and is accordingly Halldén incomplete. (Adding that formula as a new axiom gives the logic S8.) ◀

It would be interesting to know in general terms which such logics fall outside of the range for which a homophonic model theory provides a notion of validity (or semantic consequence) matching provability (or syntactic consequence) according to that logic. Do such cases, for instance, simply exhibit Halldén incompleteness or variations on that theme (including here non-uniformity)?[30] What has been suggested in this section is simply that some logics do lie outside that range, Brady and Rush's claims to the contrary notwithstanding. I say "suggested" rather than "shown" because there may be room for manoeuvre, and some tricky issues arise. Let us have a glance at them.

[30]Not that this has anything to do with uniformity, though because of potential confusion here that term is not optimal, let us notice here that often logics are proposed which are not closed under uniform substitution – a fair sample can be found at the top of p. 192 in (Humberstone 2011) – but while these would be obviously outside the range, they are also not in the spirit of Brady and Rush's discussion, whence their exclusion from our concern here, back in note 44.

5 Possible reactions

Here we consider briefly three possible responses to the findings of the preceding section. A first reaction might be to dismiss Halldén incomplete logics as marginal exceptions to the essentially correct claim of Brady and Rush that the semantical approach they dismiss as trivial because automatically available is indeed automatically available in all but these marginal cases. In reply to this, we recall from the end of Section 4 that the extent of its unavailability has by no means been shown to arise only in the fact of Halldén incompleteness – its extent was left as an open problem – and further, that Halldén incompleteness is by no means to be set aside as a marginal phenomenon. Our postscript (Section 6) aims to substantiate that reply with illustrations of how the phenomenon arises.

A second response which might be anticipated, along the lines of that mentioned in note 9, is that the discussion above completely ignores the contrast between the interpretational and the underlying metalogic which is stressed in some of the material cited in Section 2 above. Guilty as charged. But even if such a distinction can be made out and shown to salvage the suggestion of triviality or guaranteed success for the position thus bolstered, considerable interest surely still attaches to the homophonic model-theoretic project as discussed here with a straightforward undifferentiated metalogic. If some other authors – Rady and Brush, as it might be – were to make the claim that soundness and completeness were trivially assured w.r.t. such a semantics, it would be worthwhile to explore the areas in which such a claim was an overstatement, and to delimit the precise range of logics – not that any such general result has been offered here – for which the claim would be correct.

A third possible reaction will be described here, on the ultimate success of which no definitive verdict will be offered. The discussion will provide an occasion to revisit the Brady and Rush's claim of triviality for proofs of semantic completeness w.r.t. the homophonic semantics, but it will also lead us into some conceptually challenging terrain. Beginning with the triviality claim, we recall once more that after describing the soundness and completeness theorems (in this setting) as trivial, Brady and Rush say:

> For completeness, for a nontheorem N, $I(N) = T$ would fail as a result of the failure of the corresponding metalogical law, applied to totally unconstrained assignments to its variables.

Contrapositively, if we may permit ourselves such a reformulation (cf. Section 3), suppose we are dealing with an unspecified logic, $\vdash$, about which we accordingly do not know whether a formula, say for the sake of example the formula $p \to ((p \to q) \to q)$, is provable, but we are told that according to the homophonic model-theoretic semantics this formula is valid. This means that, where Σ contains (at least) the set of (universally quantified) ($\#/\underline{\#}$) conditions, we have

$$\Sigma \mathrel{\underline{\vdash}} \forall I\, (I \models p \mathrel{\underline{\to}} (I \models p \mathrel{\underline{\to}} I \models q) \mathrel{\underline{\to}} I \models q).$$

What follows the "$\forall I$" on the right of the "$\underline{\vdash}$" here is indeed an instance of, in Brady and Rush's words "the corresponding metalogical law", namely, in schematic form, $\varphi \to ((\varphi \to \psi) \to \psi)$, so the current thought is that Brady and Rush may be taking it our specific instance of this schema could only be a $\underline{\vdash}$-consequence of Σ if all instances of this schema were provable in the metalogic. Then, the thought continues, since the metalogic is supposed to coincide, in its propositional fragment, with the object logic, that could only happen if all instances of the corresponding object-linguistic schema were $\vdash$ consequences of $\varnothing$, i.e., if $\vdash p \to ((p \to q) \to q)$. Thus we reason from the validity of our formula to its provability.[31]

The reasoning is rather sketchy – Σ drops out of the story somewhat mysteriously – but let us suppose that something along these lines is correct. What would an all-gaps-filled explicit version of this argument show? Abbreviating the claim that for all $I, I \models A$ ("A is valid") to "$\mathit{Val}(A)$" it begins with the hypothesis that $\Sigma \mathrel{\underline{\vdash}} \mathit{Val}(A)$, which asserts the provability of $\mathit{Val}(A)$ in the metatheory with axioms Σ and logic $\underline{\vdash}$. For brevity we could read this as saying that A is *provably* valid. So the envisaged perfecting of the argument

[31] On the occasion mentioned in the acknowledgments at the end of this paper, Brady confirmed that this was indeed what they had in mind. On the same occasion he made a response distinct from the three canvassed here, namely that mentioned and dismissed immediately after Remark 1.1 above. In a later communication circulated to those who had been in attendance, Brady explained indicated that, on reflection he would like to observe that "the key to toil-less semantics lies in the deductive presentation of semantics, i.e. free semantics (see (Brady 2010)), which was set up in natural deduction, free of disjunctive or existential instantiation. The toil in semantics is in satisfying the requirements of formula induction in standard model-theoretic semantics, e.g. primeness, negation-completeness, existential instantiation. So, our point about the soundness and completeness being toil-less when the object and meta-logics are the same still applies to free semantics. Clearly, in the case of classical logic being used in both object and meta-language, the toil is associated with establishing negation-completeness and the existence of existential witnesses. The same sorts of things would apply for weaker logics, employed in both the object and meta-languages." I include this response here, not because I understand it – or (Brady 2010), for that matter – but for the possible edification of anyone who might.

sketched above would show that any provably valid formula is provable – where the latter reference is to object-logical provability. (To parallel the semantic characterization, one could equally well say here "is provably provable".) This would be something that could be meant by semantic completeness, and it is not threatened by the Halldén material in Section 4. The argument there was that according to the homophonic semantics we have formulas A and B one of which was true on all interpretations, but whichever it was, it wasn't provable, so there is a failure of semantic completeness. But this is not a failure of semantic completeness in the new sense just isolated. If we imagine the reasoning fully formalized, with Σ beefed up (as necessary) so as to support the argument to the existence of a valuation I'' suitably mimicking I on A and I' on B, then the best we can hope for is that:

$$\Sigma \vdash \mathit{Val}(A) \veebar \mathit{Val}(B),$$

which is to say we do not have $\Sigma \vdash \mathit{Val}(A)$ or $\Sigma \vdash \mathit{Val}(B)$. This means we do not find ourselves with a counterexample to the claim of completeness in the modified sense: we have not given any reason to think that some provably valid formula is unprovable in the object logic. Whether a response along these lines can succeed is unclear. The idea that the modal logic K, for example, is sound and complete w.r.t. a semantic theory that tells us that one of $\Box\top, \neg\Box\top$ is valid but, does tell us which one – that's an idea that would take some getting used to.

6 Postscript: Halldén completeness, then and now

This Postscript assumes familiarity with the Kripke semantics for normal modal logics, and in particular with the inductive definition of truth at a point in a model, taken as a triple $\langle W, R, V\rangle$ (on the frame $\langle W, R\rangle$). While we begin in Example 6.1 with consequence relations, when discussing modal logics we are thinking of them as sets of formulas; definitions of *modal logic* and *normal modal logic* in this setting were provided in note 22. For expository convenience the focus, below, is mainly on normal modal logics. The first time Halldén completeness arose in the literature (Halldén 1951), it was as a feature of non-normal modal logics – in particular, of C. I. Lewis's S1, S2, S3 – and there is no substitute, if one wants a clear-headed summary of the early literature on this topic, for pp. 268–272 of (Hughes and Cresswell 1968), perhaps supplemented, for appropriate 'period flavour', by a perusal of Chapter 6 and Appendices II and III of (Lewis and Langford 1959).[32] A particularly attractive feature of their treatment is the idea of a Halldén incomplete logic such as S3 as being "neutral between two conflicting views" (Hughes and Cresswell 1968, 270), in this case those embodied in the two stronger logics S7 (again non-normal) and S4 (normal); the formal underpinning of this feature is given in Proposition 6.7(*iii*) below (a well-known result). From this perspective, taken up in (Humberstone 2007b), aspects of which are echoed below, the arguably unfavourable ('something missing') connotations of the 'Halldén *incomplete*' terminology are misleading:[33] *neutral* on this or that issue may be just what one wants to remain. (The issue in (Humberstone 2007b) was realism about non-actual worlds, with an attempt give the 'modal agnosticism' of (Divers 2006) a formal embodiment.) This – perhaps rather superficial – conception of neutrality should not be taken to represent an endorsement of the 'neutral arbiter' status of logical work criticized in a paper mentioned in passing in Section 1 above (as well as in the longer note on that 'On Dummett and others' in the Appendix), (Williamson 2014), in which Williamson warms his readers up with the following (p. 212):

> A natural metametaphysical hope is that logic should be able to act as a neutral arbiter of metaphysical disputes, at least as a framework on which all parties can agree for eliciting the consequences of the rival metaphysical theories. An obvious problem for this hope is the proliferation of alternative logics, many of them motivated by metaphysical considerations.

The idea of neutrality as embodied in Halldén incompleteness was not the

[32]The same authors' (Hughes and Cresswell 1996), sometimes mistakenly thought to have supplanted the original, contains no discussion of this topic. A later channeling of the less tentative side of Lewis can be found in (Shukla 1972), mentioned in Example 4.9. In Shukla's terminology none of S1, S2, S3, their provenance notwithstanding, counts as a 'Lewis system proper'.

[33]To say nothing about the apparently hostile alternative 'Halldén unreasonable' terminology, used throughout our discussion with no such criticism intended, but simply as a reminder honouring (McKinsey 1953), though here we apply it to the disjunctions witnessing the Halldén incompleteness of a logic rather than, as McKinsey did, to the logics themselves. And, perhaps distancing himself from the suggestion that anything was amiss, recall that McKinsey said "unreasonable in the sense of Halldén" (though "according to" would have been more accurate than "in the sense of"). More on the business of thinking something is wrong with Halldén incompleteness can be found in (Schumm 1993b).

idea of logic as arbitrating between competing views but rather as a way of refraining from taking sides. The discussion to follow emphasizes themes from (Humberstone 2007b), (Humberstone 2011) – esp. pp. 862–872 – and to a lesser extent, (Humberstone 2016) (see the index entries under 'Halldén completeness'). The 'then and now' part of the title is there to record the fact that we will have at least get round to looking at Halldén's own discussion from a contemporary vantage point; the intention to supplement rather than supplant the useful retrospective provided by (Schumm 1993b). But first, an example showing that even for logics, conceived as consequence relations (with disjunction behaving $\vee$-classically), the intersection of two proper extensions does not yield Halldén incompleteness.

Example 6.1 Let us return to the case of minimal logic in and the neutrality of this logic – identified with the consequence relation $\vdash_{\mathsf{ML}}$ of Section 4 – over the interpretation of $\bot$ as a truth or a falsehood. One might, accordingly, think that $\vdash_{\mathsf{ML}}$ is the intersection of two of its proper extensions, one deciding the matter one way, by having a theorem $\bot$ and the other deciding it the other, by having (for all A) $\bot \to A$ as a theorem. This thought is not quite right, since then the 'Halldén-unreasonable' disjunction $\bot \vee (\bot \to q)$ would be provable in each of them and yet $\nvdash_{\mathsf{ML}} \bot \vee (\bot \to q)$ – since $\vdash_{\mathsf{ML}}$ does after all enjoy the disjunction property. In this setting, as one gathers from (Hawranek 1987) (with (Hawranek and Zygmunt 1981) for additional background),[34] we have to think, instead, in terms of the uniformity or cancellation condition, addressing the consequence relation involved in all its splendour, and not just the consequences of $\varnothing$ according to it. The first thought got one of the two extensions of $\vdash_{\mathsf{ML}}$ right: the smallest extension $\vdash$ of $\vdash_{\mathsf{ML}}$ such that $\vdash \bot$. But the other candidate, it got wrong: what was needed was, rather, the least extension $\vdash$ of $\vdash_{\mathsf{ML}}$ such that for all formulas C, we have $\bot \vdash C$. Abbreviating each of *Verum* and *Falsum* to its first three letters, let us call these two consequence relations $\vdash_{\mathsf{ML_{VER}}}$ and $\vdash_{\mathsf{ML_{FAL}}}$, respectively. The less obvious part of the claim that $\vdash_{\mathsf{ML}} = \vdash_{\mathsf{ML_{VER}}} \cap \vdash_{\mathsf{ML_{FAL}}}$ is the $\supseteq$-direction, and for a proof – which will be given using *classical* metalogical reasoning – we make use of the

[34]A typo in (Hawranek 1987) to watch out for: schema 9 at the base of p. 284 should be $\bot \to A$ rather than $A \to \bot$. The explanation is perhaps that one is used to writing the latter as the definition of $\neg A$. The same confusion but in the opposite direction – writing the axiom when attempting to give the definition (in Polish notation this time) – can be found in line 10 of the first page of (Sobociński 1964), when putatively quoting Prior. Less likely to cause confusion: in the middle of p. 287, for "pseudo-relatively lattice" read "relatively pseudo-complemented lattice".

(easily established) lemma that for any $\Gamma \cup \{A\}$:

$$\Gamma \vdash_{\mathsf{ML_{VER}}} A \text{ iff } \Gamma, \bot \vdash_{\mathsf{ML}} A.$$

Now, contrapositively, we need to show that if $\Gamma \nvdash_{\mathsf{ML}} A$ then either $\Gamma \nvdash_{\mathsf{ML_{VER}}} A$ or $\Gamma \nvdash_{\mathsf{ML_{FAL}}} A$. Invoking the lemma above, this amounts to showing that $\Gamma \nvdash_{\mathsf{ML}} A$ then either $\Gamma, \bot \nvdash_{\mathsf{ML}} A$ or $\Gamma \nvdash_{\mathsf{ML_{FAL}}} A$. This we show by assuming (1) $\Gamma \nvdash_{\mathsf{ML}} A$ and (2) $\Gamma, \bot \vdash_{\mathsf{ML}} A$, and inferring from (1) and (2) that $\Gamma \nvdash_{\mathsf{ML_{FAL}}} A$. In view of (1), by a Lindenbaum construction we can extend Γ to some Γ^+ with the property that $\Gamma^+ \nvdash_{\mathsf{ML}} A$ while for any proper superset Δ of Γ^+, we have $\Delta \vdash_{\mathsf{ML}} A$. In view of (2), $\bot \notin \Gamma^+$, which implies that $\Gamma^+ \nvdash_{\mathsf{ML_{FAL}}} A$, since Γ^+ is accordingly closed under the transition from $\bot$ to arbitrary C. Now $\Gamma^+ \supseteq \Gamma$, so, as we wanted, $\Gamma \nvdash_{\mathsf{ML_{FAL}}} A$. As (Hawranek 1987) points out, each of (what we are calling) $\vdash_{\mathsf{ML_{VER}}}$ and $\vdash_{\mathsf{ML_{FAL}}}$ is determined by a single matrix, in which $\bot$ is assigned a (indeed *the*) designated element and an undesignated element respectively, meaning that $\vdash_{\mathsf{ML}}$ is determined by a pair of matrices and accordingly (since it is not determined by any single matrix) has 'degree of complexity' 2. From the perspective of Kripke-style model theory (Segerberg 1968) we can work with pointed frames $\langle U, \leq, Q, u \rangle$ in which $u \in U$ and $Q \subseteq U$ is upward closed under the partial ordering $\leq$ of U, an order w.r.t. which u is a least element. In any model on such a frame, Q provides the set of points at which $\bot$ counts as true (and the model provides for each propositional variable a similarly $\leq$-closed truth-set, the connectives $\wedge, \vee, \rightarrow$ being treated as per the Kripke semantics for intuitionistic logic). A is a *semantic consequence* of Γ relative to a class of such frames just in case in any model on one of those frames, if every formula in Γ is true at the distinguished point u in the model then so is A. Then $\Gamma \vdash_{\mathsf{ML}} A$ just in case A is a semantic consequence of Γ relative to the class of all frames, $\Gamma \vdash_{\mathsf{ML_{VER}}} A$ just in case A is a semantic consequence of Γ relative to the class of all frames $\langle U, \leq, Q, u \rangle$ in which $u \in Q$, and $\Gamma \vdash_{\mathsf{ML_{FAL}}} A$ just in case A is a semantic consequence of Γ relative to the class of all frames $\langle U, \leq, Q, u \rangle$ in which $u \notin Q$. It is only this last case which is not covered in (Segerberg 1968), where logics are taken as (certain) sets of formulas rather than as consequence relations, and the set of (as we would put it) $\vdash_{\mathsf{ML}}$-consequences of the empty set is referred to as J. The closest approximation to the $\vdash_{\mathsf{ML_{FAL}}}$ case covered in (Segerberg 1968) is that of intuitionistic logic, in which the corresponding frame condition is that $Q = \varnothing$, giving us, by contrast with the case of $\vdash_{\mathsf{ML_{FAL}}}$, $\bot \rightarrow A$ (for all A) as a consequence of the empty set rather than just A as a consequence of $\bot$

(i.e., of $\{\bot\}$). As this underlines for us, $\vdash_{\mathsf{ML}_{\mathsf{FAL}}}$ does not have the 'Deduction Theorem' property. In fact, by strengthening the above condition "$u \notin Q$" to $Q = U \smallsetminus \{u\}$, we get an extension of Minimal Logic in which if $\neg A$ is defined as $A \to \bot$ we have $\vdash A \vee \neg A$ for all A: see (Humberstone 2006, 64f.). (In fact $\bot$, so interpreted is written as Ω and $\neg$ as $\neg_\Omega$ so that we can have this new constant alongside a 'nowhere true' constant $\bot$ and an $(A \to \bot)$-style negation – all of which means we have an extension, in an expanded language, of intuitionistic rather than just minimal logic.) ◀

Example 6.1 invites us to draw a couple of morals. The first is the main intention of the example, namely that the focus on Halldén incompleteness of a logic as a reflection of neutrality between extensions works well in classically based modal logics but is not all-purpose indicator of a logic's being the intersection of two of its proper extensions in this way, since minimal logic is such an intersection but is not Halldén incomplete by the disjunction property argument at the start of Example 6.1. (Similar considerations arise in (Miura 1966).)

Modal logics abound in which provability does indeed coincide with a perfectly respectable notion of validity – validity over this or that class of Kripke frames – despite these logics being Halldén incomplete, which Halldén expected would rule out this possibility. But he was writing in advance of the development of the Kripke semantics and we will revisit his original qualms after talking about, in particular normal modal logics, with that semantics in mind.

Definition 1 of (Halldén 1951, 127) defines a logic, taken as a set of formulas, to be "complete with respect to an interpretation i", if and only if that set coincides with the set of formulas which "according to i, express true propositions", but does not define what an interpretation or a proposition is or what it is to express a proposition according to an interpretation, though from Definition 2, it seems that propositions can be identified either with or via (first-order?) predicate-logical formulas. The interpretations i in play are evidently not to be confused with interpretations I of the kind in play in our discussion of Brady and Rush. In giving Halldén's Definition 2 below, I have replaced his "$(x) \centerdot fx$" notation with "$\forall x(Fx)$" (etc.). This Definition 2, quoted along with a remark on C. I. Lewis, declares an interpretation i to be a *normal* interpretation – cf. note 25 – of a logic C (construed a set of formulas), if and only if:

> for all C-formulas P and Q which are such, that (1) Each contains only one variable, (2) They have no variable in common, it holds that some monadic functions F and G, and some dyadic function H are such that (3) P expresses the proposition $\forall x(Fx)$, and Q expresses $\forall y(Gy)$, (4) $P \vee Q$ expresses $\forall x \forall y H(Fx, Gy)$, (5) If $\forall x \forall y(H(Fx, Gy))$ is true, then either $\forall x(Fx)$, or $\forall y(Gy)$ is true.
>
> It can immediately be seen that Lewis's own interpretation of the Lewis calculi is a normal interpretation of them. That conditions (3) and (4) are fulfilled is trivial. That (5) is fulfilled partly depends upon the fact that the interpretation assigned to $P \vee Q$ (...) permits the inference from $\forall x \forall y(Fx \vee Gy)$ to $\forall x(Fx) \vee \forall y(Gy)$, when F and G contain no free variables.

Now one may well be puzzled by all these universal quantifiers appearing out of thin air, and think vaguely that they are quantifying over points in a Kripke model – but of course such models had no currency when (Halldén 1951) was published. From the point of the Kripke semantics – and indeed from any semantic point of view – the talk of interpreting modal formulas in this way seems to stand in need of a clarification as to whether the interpretation is intended to march in step with the truth of the modal formula or with its validity. From the specifically Kripke-semantic perspective this contrast cross-cuts another of obvious relevance: whether we are speaking locally (truth vs. validity at a point – as explained before Definition 6.2 below or globally (truth throughout a model vs. validity on a frame). At this juncture we can do no better than display Schumm's discussion of the implicit "combination of assignments" here (from (Schumm 1993b, 198)); here α and β are the disjuncts of a candidate Halldén-unreasonable disjunction and the square-bracketed insertion is an clarificatory addition:

> Now suppose neither α nor β is valid, where α and β share no variables. This means only that there are worlds (possibly different) in frames (possibly different) at which α and β can be made false (possibly under different variable assignments). There is no guarantee, nor any evident reason why there should be, that there will be a single world in a single frame at which α and β can both be made false [at the same point] under the same variable assignment, which is what it would take for $\alpha \vee \beta$ to lack validity.

The reasoning here commented on is a contraposed version of that from the quotation from Halldén, so Schumm is suggesting that what Halldén's universal quantifiers are ranging over are potential assignments to the propositional variables or sentence letters, and what Schumm is emphasizing is a possibility provided by the Kripke semantics and required for the semantic description of modal logics in general – though in elaborating the below, we concentrate on normal modal logics for simplicity – is that these assignments themselves are sets of points (or world) which may differ in their logical properties. To spell out the kind of heterogeneity involved here we need first to define the notion used above, of validity at a point: a formula is valid at a point in a frame when it is true at that point on any model on the frame.[35] This notion of a valuation V is of the assignment to all the sentence letters at once of subsets of the universe W of a frame $\langle W, R\rangle$ to obtain a model $\langle W, R, V\rangle$ on that frame, whereas the passage from Halldén handles the effects of such a valuation piecemeal, universally quantifying over the assignment to this (propositional) variable and then to that variable,…, and of course he is not specifically thinking of the objects assigned as sets of points in some frame (or model structure, as Kripke used to put it). We can think of the remarks quoted from Schumm above as doing two things. In the first place, it tells us that a more suggestive notation than Halldén's might involve writing $\forall p$ and $\forall q$ rather $\forall x$ and $\forall y$ (though Halldén did not actually use the "$\forall$" notation) with what look like familiar individual variables. In fact, apparently independently of Halldén, though a few years later, Jerzy Łoś took worked with the corresponding notion in first-order theories – variously called O-completeness or quasi-completeness – where it is individual variables that are involved.[36] The second thing the passage from Schumm does is alert us to the fact that the set of formulas valid at any given point in a Kripke frame is indeed Halldén complete,[37] and, if we were so inclined, we could then par-

[35]We speak of truth when particular valuations – expanding frames to models – are at issue, and of validity when universally quantifying away the reference to such valuations.

[36]See (Łoś 1955), (Łoś and Suszko 1955); Suszko remarks on the relation to Halldén in later publications, such as (Suszko 1971) and (Suszko 1975). One need not consider non-logical theories to find failures of 'quasi-completeness': consider the open formula, free variables having the generality interpretation $\forall y(y = z) \vee \exists z(z \neq u)$ provable in classical predicate logic with identity though neither of the (individual!) variable-disjoint disjuncts is. One could of course insert the invisible universal quantifiers binding "z" and "u" here, but this version looks (slightly) more like the Halldén phenomenon in sentential logics without propositional quantifiers.

[37]This is something to be stressed in any introduction to the topic – as, for example, in Section 4 of (Humberstone 2005).

lay Halldén's dissatisfaction with modal logics lacking that property into a Kripke-semantical setting as an argument that frames in which different formulas are valid at different points have no place in explicating modality. But even if such an argument could be made out, modal logics in the sense currently under discussion have nothing specifically to do with modality – if that means: with necessity and possibility – but are a broad generalization aspiring to encompass many applications (such as the tense logic of 'ending time' to cite a well-known Halldén incomplete case). Even in the case of metaphysical necessity itself, as mentioned in the opening paragraph of this Postscript, a case can be made for Halldén incompleteness on hedging-one's-bets grounds – more specifically for a failure of global Halldén completeness as defined below (contrasting in this respect with the tense-logical case just mentioned). This is the line explored in (Humberstone 2007b), for instance.

Definition 6.2 A frame $\mathcal{F} = \langle W, R\rangle$ is *weakly homogeneous* if there do not exist a pair of formulas A, B for which $W = W_A \cup W_B$, where A is valid (in $\mathcal{F}$) at each point in W_A, while B is not valid at every such point, and B is valid (in $\mathcal{F}$) at each point in W_B, while A is not valid at every such point.

Remark 6.3 The word "weakly" is included here because a stronger notion of homogeneity could be introduced to describe a frame in which all points validates the same formulas. Stronger still would be the 'language-free' notion of a frame $\langle W, R\rangle$ as homogeneous when for any $x, y \in W$ there is some automorphism of the frame which maps x to y. (This last notion, taken from first-order model theory, has some currency in the Halldén completeness literature – for example in (van Benthem and Humberstone 1983) and (Kostrzycka 2015). Incidentally, a variant of Theorem 2 of (van Benthem and Humberstone 1983), tailored to the Routley–Meyer semantics for relevant logic, appears independently as Theorem 5.15 of (Routle, Meyer and Brady 1982).) ◀

Figure 1: A Weakly Homogeneous Frame

Example 6.4 The frame depicted in Figure 1 is weakly homogeneous despite its points not all validating the same formulas, since $p \to \Box p$ is valid only at

v. (The same goes for $\Box p \to p$.) But the weak homogeneity condition is still satisfied because every formula valid at u (in this frame) is also valid at v. (To see this consider the p-morphism mapping each of u, v to the sole element of the one-element reflexive frame.) ◀

Definition 6.5 A class $\mathbb{C}$ of frames is *weakly homogeneous* if there do not exist a pair of formulas A, B for which $\mathbb{C} = \mathbb{C}_A \cup \mathbb{C}_B$, where A is valid on each frame in $\mathbb{C}_A$ while B is not valid on each such frame, and B is valid on each frame in $\mathbb{C}_B$, while A is not valid on each such frame.

Halldén completeness *tout court* was officially introduced in Definition 4.1; now we introduce for comparative purposes a strengthening of that notion:[38]

Definition 6.6 A modal logic (as a set of formulas) is *globally Halldén complete* if for any pair A, B of formulas, if A, B share no sentence letters and are such that $\Box^m A \vee \Box^n B$ is provable in the logic for all $m, n \geq 0$ (the exponents indicating iteration), then either A or B is provable in the logic.

Let us write $\mathbf{L}(\mathbb{C})$ to denote the modal logic – always a normal modal logic – determined by the class of frames $\mathbb{C}$, abbreviating "$\mathbf{L}(\{\mathcal{F}\})$" to "$\mathbf{L}(\mathcal{F})$". We collect some more or less straightforward consequences of the definitions of the concepts involved in them. Note that a 'modal logic' is here understood in the set-of-formulas sense; part (i) is Corollary 6.41.6 in (Humberstone 2011) and (iii) is a famous observation appearing in both (Kripke 1965) and (Lemmon 1966).

Proposition 6.7 (i) *If $\mathcal{F}$ is a weakly homogeneous frame, then $\mathbf{L}(\mathcal{F})$ is Halldén complete.*

(ii) *If $\mathbb{C}$ is a weakly homogeneous class of frames, then $\mathbf{L}(\mathbb{C})$ is globally Halldén complete.*

(iii) *A modal logic is Halldén complete iff it is not the intersection of two modal logics properly extending it.*

[38] As mentioned in note 28 of (Humberstone 2007b), the phrase *globally Halldén complete* is used in a different sense in (Kracht 1999). Recalling the discussion in Section 4 in which Kracht used the Halldén completeness terminology for a variation on the condition of uniformity (or cancellation) for consequence relations, let us add the further clarification that when this terminology is used *sans phrase*, it is the local consequence relation associated with a logic that Kracht has in mind – see note 23, and "globally Halldén complete" is used in (Kracht 1999) for the case in which it is, instead, the global consequence relation that is involved.

(*iv*) *A normal modal logic is globally Halldén complete iff it is not the intersection of two normal modal logics properly extending it.*

Part (*iii*) is a famous observation appearing in both (Kripke 1965) and (Lemmon 1966), while part (*i*) here is Corollary 6.41.6 in (Humberstone 2011), derived as a consequence of the observation that a variable-disjoint disjunction is valid at a point in a frame just in case one of the disjuncts is. Slightly varying Example 4.3 from (Humberstone 2007b) to illustrate some of these concepts, given a frame $\mathcal{F} = \langle W, R\rangle$ call a point $x \in W$ 1-*symmetric* ("in $\mathcal{F}$", though we take this as given by the context in future) just in case for all $y \in W$ if Rxy then Ryx, and 1-*transitive* just in case for all $y, z \in W$, if Rxy and Ryz then Rxz.[39]

Example 6.8 If we are interested in the class of frames in which each point is 1-symmetric or 1-transitive or both), it is easy to see that the modal logic determined by this class of frames is the smallest normal modal logic containing the formula $(p \to \Box\Diamond p) \vee (\Box q \to \Box\Box q)$ – evidently a Halldén incomplete logic. On the other hand we may instead take an interest in the logic determined by the class of frames which are symmetric or transitive or both (i.e., in which every point is 1-symmetric or every point is 1-transitive – though here we could equally well replace the point properties with those alluded to in note 39: 2-symmetry on the one hand or 2-transitivity or 3-transitivity on the other). In this case, we find that the logic in question is the smallest normal modal logic containing all formulas

$$\Box^m(p \to \Box\Diamond p) \vee \Box^n(\Box q \to \Box\Box q),$$

with $m, n \in Nat$, and now our logic is not just Halldén incomplete but globally Halldén incomplete (not globally Halldén complete, that is). ◀

Let us observe in closing that it isn't always necessary to explicitly bolster the disjunction with arbitrarily iterated occurrences of $\Box$ in the manner of Example 6.8, though, because we can let the infrastructure of the normal modal logic in play do that work for us. Consider for instance, the logic (KT_c in one popular nomenclature due to Chellas and employed in Humberstone 2016): the smallest normal modal logic containing the formula $p \to \Box p$, determined

[39] There are also concepts in this vicinity, of 2-symmetry – mentioned in (Humberstone 2007b) – and of 2- and 3-transitivity – see (Humberstone 2016, 185 ff.), though to save perusing these sources, let's just say: work your way through the variables used in these first-order conditions.

by the class of frames in which each point bears the accessibility relation to at most itself. Rewriting this axiom as $(p \leftrightarrow \Box p) \vee \Box\bot$ we reveal its (well-known) Halldén incompleteness. We leave the interested reader to check – 'globality for free' – that in fact in this case all the formulas $\Box^m(p \leftrightarrow \Box p) \vee \Box^n\Box\bot$ are already provable in this logic without further ado (without either disjunct being provable). In KT_c the behaviour of $\Box$ is exactly what it would be if we were to take $\Box A$ as having being added to classical truth-functional logic by defining it as $\Omega \vee A$, where Ω is a sentential constant about which nothing else need be said. The reason behind this choice of notation is not to remind the reader of (the end of) Example 6.1, but rather to recall the notation of (Porte 1979), in which the necessity operator of Łukasiewicz's Ł-modal logic – *not* a normal modal logic, shown to be Halldén incomplete in (Anderson 1954) – is observed to be similarly definable using an unspecified constant Ω but this time with the *definiens* $\Omega \wedge A$.[40] Operators so introduced are evidently in the class of those called extensional (in sentence position) though not truth-functional in (Humberstone 1988), where all such connectives are observed to yield Halldén incompleteness, the disjuncts (of a suitable witness disjunction) running through alternative candidate truth-functions.[41]

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[40] (Sotirov 2008, 317), notes that this observation had appeared ten (or more) years previously in a 1967 paper in German by D. Vakarelov. (Just for the record, the bibliography of (Porte 1979) includes a publicly distributed – though not actually published – conference handout by Porte incorporating this observation and dated 1962. The corresponding possibility operator is accordingly definable by $\Omega \to A$, something observed in print even earlier, at p. 189 of Prior 1956.) More information on the 'sentential constant' approach to the Ł-modal system, and on the the connection between this and KT_c, can be found in (Font and Hájek 2002), where Ω is mostly written as L.

[41] The terminology used here to describe these cases is not ideal. It is improved in §§3.1–3.2 of (Humberstone 2011).

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Appendix: longer notes for sections 1–3

For section 1

On Dummett and others. Although here we are mainly concerned with the homophonic case, it is worth recording some contrasting reactions to non-homophonic (though still isological) model theory alluded to in Section 1. At p. 331 of (Brady 2003), we read:

> In more recent times, even the study of formal logics, not based on classical logic and seen as alternatives to classical meta-logic has been pursued using a classical logic, with little if any explanation of this, thus creating a conflict between the object- and meta-logics. This has certainly been the case with relevant logics and intuitionistic logic, with very little written on the use of these non-classical logics in setting up their own meta-theory. (But, see Meyer 1985.)

The publication date for this source is 2003 though of course there was a considerable gestation period for some of the material in (Brady 2003), so it is interesting to see a rather different view of how "very little" work has been done in the direction in question coming from (Dummett 1991, 54f.), a passage quoted in (Williamson 2011) and (Williamson 2017):

> A thoroughly pernicious principle has gained considerable popularity in recent years. It is that in formulating a semantic theory the metalanguage must have the same underlying logic as the object-language. When this principle is followed, the proponent of a non-classical logic has a perfect counter to an argument in favour of a classical law that he rejects, namely, that the argument assumes the validity of the law in the metalanguage.

Williamson is not concerned with the issue of how prevalent such work has been, but with the merits of Dummett's proposed alternative methodology: that in discussing their respective logics with each other, the parties to the discussion (the classical and intuitionistic logician, for instance) should not exploit features on which they disagree, but set up any semantic apparatus in such a way as to be insensitive to that disagreement, allowing the discussion to proceed on neutral terms.[42] This desire of Dummett's to couch the discussion in terms not contested by either party is presumably what leads him to include a condition on the treatment of sentence letters p at a node a in a Beth model for intuitionistic logic – condition (iii_b) on p. 139 of (Dummett 1977) (or p. 191 of the first edition) – to the effect that "p is verified at a or p is not verified at a", whose inclusion will strike classically backgrounded readers as problematic not because it introduces something potentially controversial but because it suddenly becomes evident that Dummett is *not* conducting the discussion in the usual 'classical meta-logic' style and explicitly imposing a condition which from that point of view would be vacuous, so that the reasoning will be acceptable to intuitionist. (This condition does not have the same unfortunate effects as Brady and Rush's condition (*) is seen to have in the present section, because Dummett is presenting Beth's far-from-homophonic model theory, but doing so while conducting the discussion using no intuitionistically contested metalogical principles. Thus Dummett often includes language such as the following – from p. 163 of Dummett 1977 – "if we wish our argument itself to be intuitionistically valid".)

Dummett's 'neutral ground' idea bears on the remarks quoted from (Priest 2006) in the main body of Section 2: "Is it to be supposed that their account of this behaviour is to be given in a way that they take to be incorrect? Clearly not. The same logic must be used in both 'object theory' and 'metatheory'." Taking "incorrect" to mean "not correct", there are two senses to bear in mind for such phrases as *correct logic.* One may have in mind a favoured account of validity w.r.t. which the logic in question is sound and complete, and by *correct* mean precisely that: both sound and complete. Alternatively, with such a notion of validity in mind one may simply mean that the logic is sound w.r.t. that notion of validity. When invited by Priest's rhetorical question as to whether someone should be expected to reason in accordance with a logic which is not correct, one is inclined to agree because one should not be

[42](Aberdein and Read 2009, 649), touch on the passage quoted from Dummett with apparent approval, and also provide some pertinent further references. (Field 2008, 111–114), also stresses Dummett's desire not to beg contested questions, though he seems more confident than Dummett that a classical metatheory achieves the required neutrality. Here is Field on p. 111: "Suppose (. . .) that an advocate of a non-classical logic for vagueness were to propose a definition of validity for vague languages in vague terms. In that case, one would presumably have to use that non-classical logic to apply the definition. Most people who advocated other logics, e.g. classical logic, would protest that they didn't know what to make of the proposed logic: that when *they* employed the definition, using their own classical reasoning, they got different results from the non-classical logician who advocated the definition. The point of a definition of validity in an effectively classical set theory that virtually all theorists can agree on is precisely to avoid such an unpleasant situation."

expected to reason unsoundly by one's own lights. But this requires taking "correct" in the second of the two senses just distinguished. There is no similar pressure to answer, as Priest does "Clearly not", if correctness is taken the first way: there is nothing wrong with being asked (for dialectical purposes) to reason according to a logic which is weaker than one that is complete according to one's favoured notion of validity, and so we cannot conclude without further ado, as Priest does here, that the *same* logic must be used at the two levels. (However, the 'further ado' is readily forthcoming in Priest's case, since the logic – classical logic – he is being expected to reason in accordance with does endorse principles, such as disjunctive syllogism, which he rejects, rather than merely being expected to refrain from availing himself of logical principles he accepts.)

Williamson's discussion of Dummett, to resume that thread, follows Dummett's own example of proceeding, next, to consider the case of quantum logic in which a homophonic treatment of conjunction and disjunction – exactly the connectives for which such a treatment was unproblematic intuitionistically and relevantly – would be an evident violation of Dummett's injunction to try to couch the semantical discussion in terms which do not immediately beg the question as to whether the contested distribution laws are correct. Dummett has long stressed (e.g., in Dummett 1976, 274f.), approvingly reporting some thoughts of Putnam's, that the use of *truth-tables* to settle difference between this pair of parties violates the insensitivity demand in its presumption that the 2^k rows (where k sentence letters are involved) exhaust the cases to be considered. (The much less enthusiastic reception given to Putnam's ideas by Kripke here is reported in pp. 192–199 of Berger 2011; see also Stairs 2016.) Isn't it odd, though, that the parties to this debate entitle their papers "Is logic empirical?" on the grounds that the facts of quantum mechanics are taken, by those returning an affirmative answer to this question, as evidence against the correctness of the $\wedge/\vee$ distribution laws? After all, even if the quantum mechanical facts had been otherwise, defenders of this answer would presumably believe they *could instead have been* as currently envisaged, so the distribution laws would not be necessarily or *a priori* correct. And this mere possibility is not something that needs to be established empirically, even if it is the empirical facts that (according to Putnam and others) bring it to our attention as an actualized possibility.

Soundness without reference to a particular axiomatization. To conclude the initial puzzles raised in Section 1 by the Brady–Rush passage, one further issue calls for comment. The question of soundness and completeness usually arises when, with a notion of validity to hand, provided by whatever semantic account is in play, one wants to compare it with provability in this or that proof-system, such as an axiomatic system, a natural deduction system, or a sequent calculus. But there is no mention, even in general terms, in the Brady–Rush discussion, of any axioms or rules, to take the axiomatic route (suggested by their talking of completeness as requiring us to show that an arbitrary nontheorem is not valid, rather than, more generally, that an unprovable sequent is not valid). Though we will have a look at how some of this would go for a concrete axiomatization of a propositional logic in Section 2, it should be acknowledged that the Brady–Rush discussion can be made sense of without the consideration of proof system: we simply take the set of (in the axiomatic case) theorems as given: this is the logic we are to deal with and to show its soundness is to show that all of them are valid, and for its completeness that no other formulas of the language in question are valid. This is the course explicitly adopted in (Bacon 2013), for example, whose treatment (p. 347) helps itself "by brute force" to a predicate *Prov* of formulas and axioms in the (non-classical) formal metatheory $Prov(A)$ for each theorem A of the object logic and (what we may think of as) the negation of $Prov(A)$ for non-theorem, as well as inductive axioms like those concerning the connectives in (Brady and Rush 2009).[43] Further, while Brady and Rush discuss arbitrary logics (concentrating on arbitrary non-classical logics but presumably not intending to exempt classical logic from the claimed triviality of the procedure they describe – see the last paragraph of Section 3), Bacon delimits algebraically a range of logics, essentially Rasiowa's 'standard systems of implicative extensional propositional calculi' (to quote the title of Rasiowa 1974, §8.5), and – a definite bonus – stresses logics as consequence relations (or suitable sets of sequents, we may say) rather than sets of formulas, for consideration in (Bacon 2013):[44] each such logic is the consequence relation

[43]Bacon writes "$Prov(\ulcorner\varphi\urcorner)$" rather than "$Prov(A)$" and has a slightly different way of dealing with the sentence letters than Brady and Rush. The full sentence in which the phrase "brute force" appears is as follows: "Here this done by brute force – a thorough metatheorist might wish to set up a classical theory of syntax and provide an axiomatic account of provability." An example of such thoroughness, devoted to the case of classical propositional logic, is perhaps provided by (Mason 1985).

[44]Throughout the discussion *consequence relations* are to be understood in accordance with the standard definition, as given for example in (Kracht 1999), §1.4; when considered as logics, we take them to be specif-

determined by which is given a matrix based a complete lattice (top element designated) equipped with a binary implication-interpreting operation connective satisfying the conditions of (Rasiowa 1974) defining what are there (Chapter 2) called implicative algebras.[45]

For section 2

Brady on the interpretational meta-logic. On turning, as instructed in the quotation in Section 2 from (Brady and Rush 2009), to Chapter 13 of (Brady 2003), one finds a section entitled 'Interpretational Meta-Logic and the Semantics of Quantified Relevance', with the following words by way of explanation of this distinction (p. 332); the first ellipsis in this quotation recalls such conditions – numbered (i), (ii), (iii) in (Brady 2003) – as those for $\rightarrow$, &, etc., under (2) of the quoted passage in Section 1:

> When one thinks about it, there is a part of the meta-theory that corresponds most closely to the object-logic, that is, the logic of the meta-logical connectives used in expressing the interpretation statements (. . .) A classical meta-logic could be built up from just these meta-logical connectives, as it is part of the Hilbertian usage of meta-logic. This meta-logic could reasonably be seen as the interpretation of the corresponding object-logic and thus we will call it the *interpretational meta-logic*. This is just the logic of the interpretation statements, without reference to the *underlying meta-logic*, which we take to be the logic used to set up the rest

ically substitution-invariant consequence relations – satisfying the condition (sub.) in Kracht's discussion, that is – and all examples in the present paper are also finitary (satisfy Kracht's condition (cmp.), that is).

[45]The logical principles concerning the implicational connective, for which we write $\rightarrow$, thereby secured are: $A, A \rightarrow B \vdash B$; $\vdash A \rightarrow A$; $A \vdash B \rightarrow A$ (evidently a principle which would not appeal to Brady and Rush, given their relevant-logical sympathies); $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$; and $A \rightarrow B, B \rightarrow A \vdash C(A) \rightarrow C(B)$ where $C(A), C(B)$ are formulas resulting from uniformly substituting resp. A, B for the same sentence letter occurring in some formula C. If that sentence letter is, say p, one would often more explicitly write C as $C(p)$, but since its identity is immaterial, the neutral notation $C(\cdot)$ is more suggestive and is used in the main body of Section 2. (See Chapter 8 of Rasiowa 1974 for details, and Font 1999 for corrections of some aspects of Chapters 1 and 8 of Rasiowa 1974.) Returning to Bacon, the binary meets and joins in the lattice interpret conjunction and disjunction connectives, with their potentially infinitary cousins interpreting quantifiers. The latter also provide top and bottom elements, the latter being used as the consequent of an implication to interpret negation (slightly oversimplifying the actual details of Bacon's presentation here, which in fact includes not only first-order quantifiers but monadic second-order quantifiers, the latter used for quantifying over models in the setting in which all we need to know about a model is which atomic formulas it verifies, and so models can be identified with sets or properties of such formulas).

> of the semantics. Moreover, for some suitable purpose, we could separate off this interpretational meta-logic from the underlying meta-logic, even though both logics are classical. Since the interpretational meta-logic satisfies (i), (ii) and (iii), it is indeed an applied classical logic, applied to interpretation statements of the form $I(A) = T$ (or F), with (i)–(iii) acting as meaning postulates which determine the interaction of the meta-logical connectives with these statements. (Note that the 'iff's in (i)–(iii) are of the underlying meta-logic.)

Although Brady supplies several further paragraphs in elaboration of this distinction, which broaches the subject of the quasi-homophonic version of the connectives in question, the idle parameters being for points in a Routley–Meyer model, the present author was able to make little better sense of them than of the above passage and the interested reader is directed to that discussion (Brady 2003, 332–334). The reason for including the above passage is the fact that Brady and Rush refer to it in (Brady and Rush 2009) as background – despite the difference in orientation ("both logics are classical"[46]) – and because of the last parenthetical sentence: the occurrences of 'iff' are part of the (language of) the underlying meta-logic.

Even if it is not be entirely clear what Brady has in mind, this passage does at least allow us to conclude that the occurrences of "iff" in the conditions the Section 1 passage need not correspond to a (primitive or defined) connective of the object language. Thus in what may seem to be cavalier disregard for the contrast for the problematic distinction between interpretational and underlying meta-logics here – if this is anything other than the contrast between homophonic and non-homophonic semantic *theories* in the single meta-logic in play – we take it that the metalanguage will typically have additional logical vocabulary governed by the (one and only) meta-logic in accord with which the semantics is being discussed, over and above the logical vocabulary of the object language, alongside the quantifiers already granted that status in Section 1 (for the semantics of a propositional logic).

Bacon and others. As was mentioned in the second of the longer notes on Section 1, (Bacon 2013) is not concerned to show the soundness of a

[46]In which case, one might ask, why are we talking about two logics? And the logics differ how do they get to rub shoulders together in reasoning to a verdict of validity or invalidity of some candidate formula or inference in the object language, for instance?

particular axiomatization of a logic within the (algebraically delimited) range with which but just with showing that all the provable formulas – however they get to be provable – are valid, where this last is a matter of truth on every interpretation much as in Brady and Rush's discussion. The validity and provability of a formula A are expressed in the metalanguage by $Prov(A)$ and $Valid(A)$.[47] He calls the semantic theory in this style for a logic L, T_L – so *Prov* and *Valid* mean, respectively, provability in L and validity according to T_L – and describes L as *weakly sound* w.r.t. T_L if (quoting from the formulation in Coro. 2.3 of Bacon 2013,[48] changing "φ" to "A") "$Valid(A)$ follows in L from $T_L \cup \{Prov(A)\}$", which in an adaptation of the notation used in the body of Section 2 would be written as

$$T_L, Prov(A) \underline{\vdash}_L Valid(A),$$

with T_L playing the same kind of ancillary role as "$(\rightarrow/\underline{\rightarrow})$" in (mp) in Section 2, and which Bacon distinguishes from *strong soundness*:

$$T_L \underline{\vdash}_L Prov(A) \underline{\rightarrow} Valid(A).$$

Note that this strengthening of soundness has nothing to do with the strengthening alluded to in the discussion of $\vdash_{BCK}$ in Section 2 which involves allowing $n > 0$ in the claim that $A_1, \ldots, A_n \vdash_{BCK} B$ implied the truth of B on every interpretation verifying each of the A_i. Bacon argues strong soundness for all L in his range, according to the corresponding T_L, on the basis of weak soundness of aspects of his T_L not gone into in this summary, rather than on the basis of the deduction theorem for L, which he notes not all logics in the range satisfy – the case of $\vdash_{BCK}$ (and thus $\underline{\vdash}_{BCK}$) considered here being one such case.[49] ((Girard and Weber 2018) also take as their basic logic *BCK*, with quantifiers added as well as multiplicative or intensional conjunction – see note 7 – but additive or extensional disjunction, on the other hand, and add modal operators with a view to treating them semantically using the

[47] Bacon actually uses letters from the range φ, ψ, puts corner-quotes around them for talking about them in the metalanguage.

[48] I quote the Corollary here rather than the actual definition(s) at the start of Section 2.3 of (Bacon 2013), since weak soundness and completeness are given each other's definitions by mistake (as I am grateful to Bacon for confirming for me).

[49] A famous counterexample here: $p \rightarrow (p \rightarrow q), p \vdash_{BCK} q$ while $p \rightarrow (p \rightarrow q) \nvdash_{BCK} p \rightarrow q$; there are of course more general 'parametrized local' versions of the deduction theorem which are satisfies in such cases, that of $\vdash_{BCK}$ being reviewed in Example 2.1.2 of (Czelakowski 2001) – or Corollary 1.29.10 in (Humberstone 2011) for a range of cases.

Kripke model-theoretic truth-definition in the modal-free part of an analogous metalogic, paying special attention to the complications caused by the lack of *contraction* for this enterprise.) While on the subject of soundness, mention should be made of (Weber, Badia and Girard 2016), in which a *paraconsistent* metalogic is used in the semantic metatheory, the latter theory itself being *inconsistent* – and proudly so – with the authors establishing to their own satisfaction at least (p. 542) that the object logic is sound and not sound (as well as that the law of excluded middle is both valid and invalid). Hmm...

For section 3

Bell and others. What follows is a passage from p. 6f. of (Bell 1988), discussing in a free and easy way, which is nevertheless insistently intuitionistic in its logic, an aspect of the models ('smooth worlds') of the (smooth) infinitesimal analysis that the book – to which the attention of the philosophy-of-logic community has recently been drawn by (Shapiro 2014) – is all about; in the passage quoted $\mathbb{S}$ stands for such a 'smooth world', and the use of I (for Infinitesimals) has nothing to do with its use elsewhere in our discussion for interpretations:

> If we now call two points a, b on the real line *distinguishable* or *distinct* when they are not identical, i.e., *not* $a = b$ – which as usual we write as $a \neq b$ – and indistinguishable in the contrary case, i.e., if not $a \neq b$, then in $\mathbb{S}$, indistinguishability of points will not imply their identity. As a result, the 'infinitesimal neighbourhood of 0' comprising all points indistinguishable from 0 – which we will denote by I – will, in $\mathbb{S}$, be nonpunctiform in the sense that it does not reduce to $\{0\}$, that is,
>
> it is not the case that 0 is the sole member of I.
>
> If we call the members of I *infinitesimals*, then this statement may be rephrased:
>
> it is not the case that all infinitesimals coincide with 0.
>
> Observe, however, that we cannot go on from this to infer that
>
> there exists an infinitesimal which is $\neq 0$.

> For such an entity would possess the property of being both distinguishable and indistinguishable from 0, which is clearly impossible.

If being the sole member of I, as in the first line inset here, is being the only thing that is an element of I then the three representations canvassed in the main body of Section 3 for "Only Fs are G" would be, in order: (i) $\forall x(x \in I \rightarrow x = 0)$, (ii) $\forall x(\neg x = 0 \rightarrow \neg x \in I)$, and (iii) $\neg\exists x(x \in I \wedge \neg x = 0)$. It seems clear from the gloss Bell gives in the following inset as a rephrasing follows the structure of the negation of (i), and presumably more generally takes "$a \in \{b_1, \ldots, b_n\}$" to amount to "$a = b_1 \vee \ldots \vee a = b_n$". Compare now, (Bacon 2013), discussing a logic for vagueness which is like intuitionistic logic in at least this respect: it does not endorse the law of excluded middle.[50] At p. 342 there Bacon has us consider a function v, defined in deliberately vague by saying that $v(P_i) = 1$ if i is small and $v(P_i) = 0$ otherwise, adding that "v here is 'bivalent' in the sense that the codomain of v consists of two truth values: $\{0, 1\}$." To this is appended a footnote (note 7) which reads

> To say that D is the codomain of a function f is to say that $\forall x(\exists y(f(y) = x \rightarrow x \in D))$. In the logics considered the codomain of v being $\{0, 1\}$ does not entail that $v(x) = 1 \vee v(x) = 0$ for any x.

Presumably this last means (eliminating an ambiguity in "any" here): "does not entail that for all x $v(x) = 1 \vee v(x) = 0$", though the significance of writing "$\forall x(\exists y(f(y) = x \rightarrow x \in D))$" rather than simply "$\forall y(f(y) \in D)$" is not immediately clear,[51] but taking this simpler formulation as adequate the attention may be to say that, recalling that D is, in the case under consideration, $\{0, 1\}$, $y \in \{0, 1\}$ does not mean, as suggested above for Bell, $y = 0 \vee y = 1$ but rather $\neg(y \neq a \wedge \neg y \neq b \wedge \neg y \neq c)$. Bacon himself may not quite have intuitionistic logic in mind for vagueness, though the suggestion is occasionally made (e.g., Putnam 1983) and this gives a weaker notion of set membership than that we presumed was operative in Bell's discussion, serving again to

[50] On p. 104 in a chapter of (Bell 1988) making the logic explicit, the status of the law of excluded middle in the theory of smooth infinitesimal analysis is explained in more detail by Bell: (i) the theory proves the negations of some universally quantified 'excluded middles', rather than simply failing to prove the quantified forms, and (ii) in "most models" of the theory all disjunctions $A \vee {\sim}A$ are true when A is a *closed* formula.

[51] Bacon has offered (p.c.) some clarification: function symbols are not officially part of the language he is using here and "$f(x) = y$" is shorthand for "$\langle x, y\rangle \in f$", with "$f(x) \in \{0, 1\}$"" short for something along the lines of like "$\exists y((x, y) \in f \wedge y \in \{0, 1\})$" or "$\forall y((x, y) \in f \rightarrow y \in \{0, 1\})$".

illustrate the challenging multiplicity of interpretations weakening one's logic can induce, to say nothing of the awkwardness of having to figure out exactly when a transition from $\neg A \rightarrow B$ to $A \vee B$ and thence to some further conclusion can be reformulated (still terminating at that further conclusion) so as to keep one's reasoning acceptable by the lights of intuitionistic logic, and similarly with the case of the converse transition in the case of relevant logic.[52]

[52]This leads us in Section 4 to explore options – such as classically based modal logics – that retain classical (non-modal) logic, in the interests of making things easier to think through, rather than out of some conviction of the all-purpose superiority (cf. Gabbay 1993). And although it is sometimes hard to say when this or that classical principle is essential to a piece of reasoning, it is not always hard: see the index entries under "choice of meta-logic" in (Humberstone 2011), for instance, as well as such discussions as those of (Bell 1988).

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RESEARCH ARTICLE

C.I. Lewis, E.J. Nelson, and the Modern Origins of Connexive Logic

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Abstract: Modern logic owes an important debt to C.I. Lewis and his students. In addition to Lewis's five modal logics, they are responsible for the creation (or discovery) of the logic of analytic implication and connexive logic. In this paper, we examine E.J. Nelson's connexive logic as an attempt to formalise the notion of entailment while avoiding the paradoxes of strict implication. We also look briefly at the reception of Nelson's logic and at Lewis's reply to it.

Keywords: Non-classical logic; C. I. Lewis; E. J. Nelson; relevant logic; paraconsistent logic; modal logic.

1 Introduction

In this paper, we explore the relationship between the formulation of a connexive logic in the early work of E.J. Nelson (as illustrated in Nelson 1929, 1930, 1933, 1936a) and C.I. Lewis's construction and understanding of his logic of strict implication. Before we begin to explore Nelson and Lewis's logical

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views, here is some historical background. In the late 1920s and throughout the 1930s, a small school of logic students formed around C.I. Lewis in the Harvard Philosophy Department. The best known members of this school were William Parry, E.J. Nelson, and Daniel Bronstein, all of whom wrote theses on the nature of implication or entailment. Nelson (1900-1988), who is the subject of this essay, was from Washington State. Before coming to Harvard, he earned a bachelor's and master's degrees at the University of Washington. He studied at Harvard from 1925 to 1929, where he earned his PhD. He then was granted a travelling scholarship by Harvard, which he took at Göttingen and the Sorbonne, before taking up a teaching position at the University of Washington. He taught there until 1952, when he moved to Ohio State University, where he taught until his retirement in 1968.[1]

Connexive logic is characterised as having some or all of the following principles. Aristotle's theses state that no proposition entails its own negation:

$$\neg(A \to \neg A)$$
$$\neg(\neg A \to A)$$

And Boethius's theses state that no proposition entails both a proposition and its negation:

$$(\mathrm{A}\to B) \to \neg(A \to \neg B)$$
$$(\mathrm{A}\to \neg B) \to \neg(A \to B)$$

As the names of these theses suggest, connexive logic has its roots in ancient philosophy. These theses were adopted and logics contained them were developed by mediaeval logicians such as Abelard.[2]

In his PhD thesis and elsewhere, Nelson motivates his approach to logic by claiming that it avoids the paradoxes of strict implication, namely: that every proposition entails every logical truth and that a contradiction entails every proposition. Lewis famously gives an argument for the acceptance of the latter paradox in *Symbolic Logic*. Perhaps less known is his argument for the same paradox in his 1918 book, *A Survey of Symbolic Logic*. It is the argument in the *Survey* to which Nelson reacts, and so we examine it in section 2 below.

[1]The information here was obtained from a biography of Nelson found at `https://www.pdcnet.org/apapa/content/apapa-2013\relax$\@@underline{\hbox{\}}\mathsurround\z@$\relax0347\relax$\@@underline{\hbox{\}}\mathsurround\z@$\relax0348\relax$\@@underline{\hbox{\}}\mathsurround\z@$\relax547?file{\}type=pdf`.

[2]For general histories of connexive logic, see (McCall 2012) and (Wansing 2016).

Nelson's logic, as we shall argue, is not just a reaction against the adoption of the paradoxes of strict implication by his supervisor, C.I. Lewis, but also an attempt to recast Lewis's project of formulating a logic of entailment. For Nelson, '$A \rightarrow B$' expresses that B is deducible from A in some intuitive sense of 'deducibility'. If we take the arrow to mean strict implication, this is true for Lewis as well.

Nelson follows Lewis by beginning with a notion of consistency and defining an entailment connective in terms of it and negation. Nelson's consistency connective, however, is more intensional than Lewis's, and its use results in a very different logic of entailment. Overall, Nelson's treatment of the connectives is far more intensional than Lewis's. His treatment of conjunction, in particular, is intensional. He rejects the thesis of simplification, $(A \wedge B) \rightarrow A$, even though he admits that if $A \wedge B$ is true then A must be true. His approach to entailment, and logic more generally, is not truth-conditional in any ordinary sense. As we shall see, the notions of intension and meaning are of central importance to Nelson's approach to logic and motivate the system that he accepts.

Although Lewis rejects Nelson's logic, he does come to accept a philosophy of logic that is very intensional in the manner of Nelson's philosophical views. They come to use very similar language to describe what logic is about.

In this paper we examine the system that Nelson puts forward in his PhD thesis and in (Nelson 1930).[3] We think that examining his actual system sheds a lot of light on his thinking behind his advocacy of connexive logic.

2 The early argument for explosion[4]

In chapter V, section V of the *Survey of Symbolic Logic*, Lewis states that two paradoxes of strict implication in fact are theorems of his system (Lewis 1918, 335):

3.52 $\sim p \prec (p \prec q)$;

[3]The system in the thesis seems to have a rule of adjunction (which is strangely formulated) but this does not appear in (Nelson 1930).

[4]We refer to this argument as "early" because it is presented 14 years earlier than his more famous argument of (Lewis 1951)

3.55 $\sim\neg p \dashv (q \dashv p)$.

In (Lewis 1918), "$\sim$" means "it is impossible that". So, 3.52, says that if p is impossible, then it entails everything. "$\sim\neg p$" means "p is necessary", so 3.55 means that if p is necessary, then it is entailed by every proposition.

The argument that Lewis gives in section V, however, does not prove either 3.52 or 3.55. Rather, it justifies the thesis form of explosion (the proposition that a contradiction entails everything, as opposed to a rule form according to which contradictory theorems entail that everything is a theorem). The argument begins with the following three principles.

P1 $(p \wedge q) \dashv q$

P2 $((p \wedge q) \dashv r) \dashv ((p \wedge \neg r) \dashv \neg q)$

P3 $(((p \wedge q) \dashv r) \wedge (r \dashv s)) \dashv ((p \wedge q) \dashv s)$

P1 is just the principle of simplification. We discuss this at some length below. P2 is the principle of antilogism. As we shall see presently, it is a logical principle that Lewis and his school took to be a central law of logical reasoning. P3 is just an instance of Lewis's transitivity axiom. Lewis also appeals to uniform substitution, the replacement of equivalents, and a rule form of transitivity of $\dashv$.

Here is Lewis's argument as presented in (Lewis 1918, 336):

1.	$(((p \wedge q) \dashv q) \wedge (q \dashv r)) \dashv ((p \wedge q) \dashv r)$	P3
2.	$((p \wedge q) \dashv r) \dashv ((q \wedge \neg r) \dashv \neg p)$	P2
3.	$(((p \wedge q) \dashv q) \wedge (q \dashv r)) \dashv ((q \wedge \neg r) \dashv \neg p)$	1, 2, transitivity rule

By substituting q for r we obtain:

$$(((p \wedge q) \dashv q) \wedge (q \dashv q)) \dashv ((q \wedge \neg q) \dashv \neg p)$$

By P1, $(p \wedge q) \dashv q$ is a theorem. And Lewis assumes that $q \dashv q$ is a theorem. Hence, by the rule of adjunction, $(((p \wedge q) \dashv q) \wedge (q \dashv q))$ is a theorem, and so, by modus ponens,

$$(q \wedge \neg q) \dashv \neg p$$

is provable. As Lewis says "p itself may be negative" and so "this impossible proposition implies anything" (ibid.). This last step appeals to the equivalence of a proposition and its double negation and the substitutivity of propositions and their logical equivalents.[5]

In his early work (Nelson 1929; 1930), Nelson rejects P1 and modifies P3 in his logic. He discusses P2 at length, both in his thesis and in (Nelson 1933), but he retains it. As we have said, the principle of antilogism is taken by Lewis to be a central principle of deductive inference. The historical reasons for this have to do with the work of Christine Ladd-Franklin (1983), who reconstructed traditional syllogisms as "inconsistent triads" of propositions. For example, Barbara is of course

$$\frac{\begin{array}{c}\text{Every } A \text{ is } B\\ \text{Every } B \text{ is } C\end{array}}{\therefore\ \text{Every } A \text{ is } C}$$

Construed as an inconsistent triad it becomes

$$\begin{array}{l}\text{Every } A \text{ is } B\\ \text{Every } B \text{ is } C\\ \text{Some } A \text{ is not } C\end{array}$$

Any two of these propositions entails the negation of the third. Hence, the valid syllogisms that can be generated are:

- Every A is B, every B is C $\therefore$ every A is C;
- Every A is B, some A is not C $\therefore$ some B is not C;
- Every B is C, some A is not C $\therefore$ some A is not B.

Lewis cites Ladd-Franklin's reasoning about syllogisms with approval both in the *Survey* (Lewis 1918, 108–110) and in *Symbolic Logic* (Lewis and Langford 1951, 60–61). Nelson seems to have followed him in thus treating the antilogism as a key deductive principle.

[5] Given these assumptions, the argument can be made much simpler:

1.	$(q \wedge \neg p) \strictif q$	P1, US
2.	$(q \wedge \neg q) \strictif \neg\neg p$	1, P2, MP
3.	$(q \wedge \neg q) \strictif p$	2, DN, Eq

Antilogism has a mixed reception in Lewis's school. Nelson considers modifying it in (Nelson 1933). William Parry – a student of Lewis's in the early 1930s – criticises antilogism because it clashes with the key idea behind his logic of *analytic implication*. On Parry's view, no implication $A \to B$ is a theorem unless all the propositional variables in B occur in A. But, if we accept antilogism, we may derive implications that disrespect Parry's principle from implications that respect it. For example, $(A \wedge B) \to C$ may be such that $Var(C) \subseteq Var(A) \cup Var(B)$, but this is no guarantee that $Var(\neg B) \subseteq Var(A) \cup Var(\neg C)$ in $(A \wedge \neg C) \to \neg B$.

3 Consistency and entailment

Nelson's approach to logic is extremely intensional. Nelson not only interprets the entailment connective as an intensional operator, but he also interprets conjunction intensionally:

> I do not take $p \wedge q$ to mean "p is true and q is true", but simply "p and q", which is a unit or whole, not simply an aggregate, and expresses the joint force of p and q. (Nelson 1930, 444)

How to understand the notion of the "joint force" of p and q is unclear. Nelson postulates very few axioms for conjunction in his logic (see §4), so we know very little about its nature.

Nelson uses his intensional conjunction to reject the thesis of simplification $((p \wedge q) \to p)$:

> Naturally, in view of the fact that a conjunction must function as a unity, it cannot be asserted that the conjunction of p and q entails p, for q may be totally irrelevant to and independent of p, in which case p and q do not entail p, but it is only p that entails p. I can see no reason for saying that p and q entail p, when p alone does and q is irrelevant, and hence does not function as a premise in the entailing. (Nelson 1930, 447)

In this passage, Nelson alludes to some strong form of *real use* criterion for premises in an inference. The proposition $p \wedge q$ can only entail r when both p

and q are really used in the derivation. As we shall see in section 4 below, it would seem that Nelson's notion of real use is very strict indeed.

It is interesting to compare Nelson's conjunction with the intensional conjunction of relevant and linear logic. The intensional conjunction of relevant and linear logic is called "fusion". Here we represent it by $\otimes$. Like Nelson's conjunction, fusion does not satisfy simplification, that is, $(p \otimes q) \to p$ is not valid in relevant or linear logics. Nelson's connective, however, satisfies the principle of idempotence:

$$p \to (p \wedge p)$$

In strong relevant logics like the the logic R of relevant implication or E of relevant entailment, fusion is idempotent. But it is not in linear logic or in weaker relevant logics. Idempotence $(p \to (p \otimes p))$ in these logics is equivalent to contraction:

$$(p \to (p \to q)) \to (p \to q)$$

Moreover, adding contraction to these logics, by Curry's paradox, trivialises naïve theories of truth, sets, or properties.

Nelson observes that the idempotency of intensional conjunction, together with antilogism, yields undesirable outcomes. Tweaking Nelson's example slightly, "Rome is in Italy" entails "Rome is in Europe". However, if "Rome is in Italy" is equivalent to "Rome is in Italy and Rome is in Italy", then the latter also entails "Rome is in Europe", whence antilogism gives that "Rome is in Italy and Rome is not in Europe" entails "Rome is not in Italy". This, however, is absurd, because our conclusion does not follow from the "joint force" of the conjuncts. Nelson, however, does not go on to question idempotency of his intensional conjunction. Rather, he seems to suggest that "Rome is in Italy and Rome is in Italy" is actually a *simple* sentence, and not a conjunction of two simple sentences:

> If at any place in a symbolic statement, an additional mark does not add to the logical significance, that mark should be dropped out. [...] We might formulate this as a principle of significance: $p \wedge p$ must be given no significance, either in operations or in inferences, beyond that possessed by p. (Nelson 1933, 281)

Given his view that '$p \wedge p$' says no more than 'p', conjunction must be idempotent.

In the mid-1930s, however, Nelson's opposition to simplification begins to waver. He says that simplification is more intuitive than addition ($p \rightarrow (p \vee q)$), and suggests that addition may be abandoned and that either uniform substitution be restricted or that antilogism be restricted with a consistency proviso about the sentences at issue (Nelson 1933, 279).[6] We think that he may have been influenced in this by Parry. Parry's analytic implication severely restricts addition, in an attempt to avoid the paradoxes of strict implication. Parry's thesis was defended in 1931, and so its ideas and results may not have been known to Nelson until well after he wrote his own dissertation and (Nelson 1930).

In this paper, we concentrate on Nelson's entailment and consistency operators as they are developed in (Nelson 1929; 1930). These two closely linked connectives are the central concern of most of Nelson's writings about connexive logic. In addition, Nelson's critique of Lewis concentrates on the latter's treatment of these two connectives.

First, let us look at Lewis. In the *Survey*, Lewis defines the consistency connective ($\circ$) as follows:

$$A \circ B \;=_{df}\; \neg \sim (A \wedge B)$$

But we interpret this definition in terms of possibility and standard (extensional) negation:

$$A \circ B \;=_{df}\; \Diamond(A \wedge B)$$

In the *Survey*, Lewis understands possibility in syntactic terms. He defines a "possible situation" as a consistent set of propositions. An "impossible situation" is an inconsistent set of propositions (Lewis 1918, 333). Then, $A \circ B$ is true if and only if there is some possible situation that contains both A and B.

Strict implication is defined in terms of consistency and extensional negation:

$$A \prec B \;=_{df}\; \neg(A \circ \neg B)$$

Lewis interprets strict implication in terms of situations by saying that $p \prec q$ is true if and only if any situation in which p is true and q is false is impossible (ibid.). Lewis's treatment of possibility, consistency, and strict implication is

[6]In a letter to the editors of *Mind* in 1936, Nelson says "I still believe, however, that $(p \wedge \neg p) \rightarrow q$ is unacceptable, but that it can be avoided by means other than the rejection of $(p \wedge q) \rightarrow p$" (Nelson 1936b).

not semantic in the modern sense. Rather these notions are understood in terms of sets of propositions from which we can or cannot derive contradictions. The logic used to determine what sets are consistent is the logic of strict implication itself (see Lewis 1923a). Whether the sort of circularity involved here is vicious is not our current concern. Rather, we just wish to indicate the ways in which Nelson's view is different from Lewis's.

Nelson's approach to consistency and inconsistency is much less formal than Lewis's. We think that Nelson's notion of inconsistency is more accurately represented as a relation of *conflict* between meanings, rather than consistency in any standard sense.[7] For the most part, Nelson thinks that we need to examine the meanings of sentences to determine whether they are in conflict with one another. This is an informal procedure, in the sense that it cannot be fully represented in a formal system. Nelson, however, does give us some structural constraints on this notion of conflict, and these constraints are represented formally. Here are three of the most important constraints:

1. No proposition conflicts with itself.

2. Every proposition conflicts with its own negation.

3. No proposition entails any proposition with which it conflicts.

Nelson uses Lewis's notation ($\circ$) for consistency (i.e. failure of conflict). He defines conflict ($|$) as

$$A|B \ =_{df} \ \neg(A \circ B)$$

and he defines his entailment connective in terms of conflict. We use '$\rightarrow$' for Nelson's entailment. He used 'E', but we find that rather ugly. Here is the definition of entailment:

$$A \rightarrow B \ =_{df} \ A|\neg B$$

The second constraint is an axiom of Nelson's logic:

$$p \rightarrow p$$

[7]Nelson's glosses on consistency and conflict are rather enigmatic. For example (Nelson 1936a, 507): "even if p, possibly $\neg q$" is a reading of '$p \circ \neg q$'. The use of 'even if' and the obvious scope problems concerning 'possibly' in this formulation obscure its meaning. We think the notion of conflict and lack of conflict make more sense of $|$ and $\circ$.

i.e. $p|\neg p$. The third constraint is also an axiom:

$$(p \to q) \to (p \circ q)$$

By uniform substitution and modus ponens, we then can derive the first constraint:

$$p \circ p$$

or, in terms of conflict (using double negation introduction):

$$\neg(p|p)$$

Thus no proposition conflicts with itself. Even contradictions fail to conflict with themselves. Nelson's notion of conflict is not truth conditional, unlike Lewis's notion of consistency. On Lewis's view, as we have seen, two propositions are consistent if and only if they can be in the same possible situation. A contradiction cannot be in a possible situation. The notion of conflict has to do with oppositions between the contents of propositions.

Graham Priest has suggested that negation in connexive logic be interpreted as "cancellation" negation (Priest 1999). On this view, contradictions are taken to have no content. We do not think that Nelson interpreted his negation in this way (although this is not to say that this might be a better way to interpret his negation). Contradictions, like all other statements, express propositions, and propositions are taken to have positive contents, in some sort of Fregean sense. Nelson does not consider the amount of content a statement has. He has a purely qualitative understanding of content.[8]

The difference in approach to contradictions/conflicts is part of a basic difference in philosophy of logics between Nelson and Lewis. In the 1920s, Lewis adopts a pragmatic approach to analyticity, apriority, and logical truth (Lewis 1923b; 1930; 1929). Part of this pragmatic conception is that analyticity and logical truth are conventional. In the early 1920s, Lewis develops his earlier notion of situations into what he calls *systems*. Systems are logically closed sets of states of affairs (Lewis 1923a).[9] On the pragmatic view, which sets of states of affairs count as systems is to a large extent a matter of convention.

[8]Priest's reading of connexive logic is rather post-Carnap/Bar Hillel. It treats the semantics of sentences in terms of how much information they convey. Priest's interpretation of connexive logic may be more fruitful, but our purpose here is purely historical and we only wish to examine what Lewis and Nelson are doing.

[9]Lewis uses the terminology "facts" for what we call "states of affairs". The reason we do not adopt his terminology is that "fact" now generally refers to entities that are true whereas states of affairs need not be true.

For Nelson, on the other hand, which intensional relations (such as conflict) hold between propositions is evident to one in an act of immediate intuition, which is independent of “any acquaintance with particulars” (Nelson 1929, 3). Thus, whereas Lewis has a pragmatic and conventionalist approach to the philosophy of logic, Nelson takes a hard realist-platonist approach.

Before we move on to our presentation of Nelson's logic, we indulge ourselves in a bit of speculation. In an appendix to *Symbolic Logic*, Lewis formulates four logics in addition to his favourite system of strict implication, which at that time is S2. He wishes to avoid committing himself to the following strong form of transitivity of $\prec$ (Lewis and Langford 1951, 496):

$$(p \prec q) \prec ((q \prec r) \prec (p \prec r))$$

When writing the first edition of *Symbolic Logic* in the early 1930s, Lewis has no proof that this form of transitivity cannot be derived in S2. He knows that it can be derived in S3. He also can prove that it does not follow in S1, which results from dropping the axiom

$$(*)\ \Diamond(p \wedge q) \prec \Diamond p$$

from S2. Lewis suggests that S1 be adopted if it is found that S2 contains strong transitivity. The axiom $*$, however, is rather intuitive and dropping it has seemed to most logicians a bad price to pay to avoid strong transitivity. The intuitiveness of $*$ rests on the reading of $\Diamond$ as possibility or as consistency (which is the same thing for Lewis). If, on the other hand, we read $\Diamond(p \wedge q)$ as saying that p does not conflict with q we may not think that $*$ is valid. For the content of p and q may not conflict, but p may conflict with itself. It may be, then, that Nelson had some influence on Lewis in the latter's formulation of S1. Whether this is really the case, we have no strong evidence.

4 Nelson's logic

In this section, we present Nelson's connexive logic, which we call ‘NL’.

We change Nelson's notation slightly, using $\wedge$ instead of mere concatenation, $\rightarrow$ instead of E, and $\neg$ instead of $-$. Nelson's primitive connectives are

$\neg$, $\circ$, and $\wedge$. Inconsistency is defined, quite reasonably, using consistency and negation:

$$A|B \;=_{df}\; \neg(A \circ B)$$

and entailment is defined in terms of inconsistency:

$$A \rightarrow B \;=_{df}\; A|\neg B$$

Intensional logical equivalence or the identity of meaning, =, is defined as usual:

$$A = B \;=_{df}\; (A \rightarrow B) \wedge (B \rightarrow A)$$

Nelson also defines an "intensional sum" (an intensional disjunction), $\bigvee$:

$$A \bigvee B \;=_{df}\; \neg A|\neg B$$

We do not discuss intensional sum in what follows, and it does not appear in any of his axioms. In the axiomatisation of his logic Nelson does use an unusual abbrevation, $A \neq B \neq C$. It is defined as follows:

$$A \neq B \neq C \;=_{df}\; (A \neq B) \wedge (B \neq C) \wedge (A \neq C).$$

Nelson set out his system on pages 138-141 of his thesis.

Axioms of NL

1.1 $p \rightarrow p$
1.2 $(p|q) \rightarrow (q|p)$
1.3 $p \rightarrow \neg\neg p$
1.4 $(p \rightarrow q) \rightarrow (p \circ q)$
1.5 $(p \neq q \neq r) \rightarrow (((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r))$
1.6 $(p \wedge q) = (q \wedge p)$
1.7 $((p \wedge q) \rightarrow r) \rightarrow ((p \wedge \neg r) \rightarrow \neg q)$

We have a few comments to make on the axiom list before we go on to state the rules of NL.

Axiom 1.1, stated in terms of the conflict connective, is

$$p|\neg p,$$

which merely states that every proposition conflicts with its own negation. Axiom 1.2 states that conflicts are symmetrical. 1.3 is not justified in terms of the notion of conflict, but rather in terms of an intuition about negation. As we shall seen below, the converse of 1.3 is also provable and the meaning of any proposition according to this logic is the same as the meaning of its double negation.

Axiom 1.4 is the only contra-classical axiom in the logic. We discussed it in section 3 above. It says that if either of two propositions entails the other, then they do not conflict with one another. Together with 1.1, 1.4 entails that no proposition conflicts with itself.

Nelson prefixes his transitivity axiom (1.5) with $p \neq q \neq r$. If we eliminate the prefix, then we get theorems like

$$((p \to p) \wedge (p \to q)) \to (p \to q),$$

that are instances of the simplification scheme that Nelson rejects. Daniel Bronstein (another of Lewis's students) accuses Nelson of committing a fallacy here (Bronstein 1936, 167–168). Bronstein points out that just because unconstrained transitivity allows the derivation of some instances of simplification does not mean that it allows the derivation of all instances. In particular, it is not shown that unrestricted transitivity will allow the derivation of any paradoxes. Moreover, one might wonder, however, whether this thesis is as objectionable as other cases of simplification, since we can think of $p \to p$ being used (rather inefficiently) to derive $p \to q$ from itself by transitivity. That is, we do not have to think of the derivation of $p \to q$ from $(p \to p) \wedge (p \to q)$ merely as a dropping of the first conjunct of the premise.

The inclusion of the prefix in this axiom[10] may indicate that Nelson has in mind a very strong form of *real-use* principle. The above instance of transitivity might be thought to be objectionable because $p \to p$ is not needed (although it may be used) to derive $p \to q$. This very strong version of the principle says that $(A_1 \wedge ... \wedge A_n) \to B$ can be allowed as a theorem only if there is no proper subset $\{i, ..., j\}$ of $\{1, ..., n\}$ such that $(A_i \wedge ... \wedge A_j) \to B$ is also a theorem. This is a very strong requirement indeed, and we do not have enough evidence to attribute it to Nelson.

The antilogism axiom (1.7) seems even more natural if read in terms of

[10] Which he continues to accept in (Nelson 1933).

conflict (modulo some fiddling with double negations):

$$((p \wedge q)|r) \to ((p \wedge r)|q)$$

This is very close to Ladd-Franklin's formulation of her principle of inconsistent triads.

NL does not contain an axiom, such as, $(p \wedge (q \wedge r)) \to ((p \wedge q) \wedge r)$, that states that conjunction is associative. His use of strings of conjunctions without parentheses makes it seem as though he thought of conjunction as associative, but it is difficult to say whether the omission of an associativity axiom was an oversight or is intentional.

Rules

The rules that Nelson states in his axiomatisation are the following:

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \qquad \frac{\vdash A \quad \vdash B}{\vdash A \wedge B}$$

We abbreviate 'modus ponens' as 'MP' and 'adjunction' as 'Adj'.[11]

In his proofs of theorems, Nelson also makes free use of two substitution rules: a standard rule of uniform substitution for propositional variables ('US') and a rule of substitution for logical equivalents. This latter rule, for which we adopt the name 'Eq' from (Hughes and Cresswell 1968), is the following:

$$\frac{\vdash A = B}{\vdash C = C'}$$

where C' results from replacing one or more occurrences of A with B in C.

NL, as we think of it, then contains axioms 1.1-1.7, the rules MP, Adj, US, and Eq.

[11]Nelson's statement of Adj is somewhat convoluted: "If a proposition, or all the propositions of a logical product, are asserted, separately, or otherwise, but as such, then whatever is entailed by or is identical with that proposition, or that logical product, as the case may be, is categorically asserted" (Nelson 1929, 141). Clearly this statement includes Adj, but also a rule that says that whatever is entailed by a theorem is a theorem. This is included in the rule MP, so we do not need to state it independently.

Here is a proof of the left to right half of the principle of double negation (DN) ($A \to \neg\neg A$) (taken, in effect, from Nelson 1929, 159):

$$\begin{array}{lll} 1. & \neg A \to \neg A & 1.1, \text{ uniform substitution} \\ 2. & \neg A | \neg\neg A & 1, \text{ def. of } \to \\ 3. & \neg\neg A | \neg A & 2, 1.2, \text{ MP} \\ 4. & \neg\neg A \to A & 3, \text{ def. of } \to \end{array}$$

Together with axiom 1.3 and Adj, we obtain

$$\vdash p = \neg\neg p.$$

We call this 'DN' and we use it extensively in our other proofs. For example, here is a proof of contraposition:

$$\begin{array}{lll} 1. & (p|\neg q) \to (\neg q|p) & 1.2, \text{ US} \\ 2. & (p|\neg q) \to (\neg q|\neg\neg p) & 1, \text{ DN, Eq} \\ 3. & (p \to q) \to (\neg q \to \neg p) & 2, \text{ def. of } \to, \text{ Eq} \end{array}$$

5 Nelson's logic as a connexive logic

A connexive logic is characterised as a system which accepts some or all of Aristotle's and Boethius's theses. Nelson's axiom 1.4 is

$$(p \to q) \to (p \circ q)$$

By the definition of conflict, DN, and Eq, we obtain

$$(p \to q) \to \neg(p|q)$$

and so, by DN, the definition of implication, and Eq we get

$$(p \to q) \to \neg(p \to \neg q).$$

This is Boethius's first thesis. In fact, 1.4 is equivalent to Boethius's first thesis:

$$\begin{array}{lll} 1. & (p \to q) \to \neg(p \to \neg q) & \text{Boethius} \\ 2. & (p \to q) \to \neg(p|q) & 1, \text{ DN, Eq, def. of} \} \to \\ 3. & (p \to q) \to (p \circ q) & 2, \text{ def. of } \circ \end{array}$$

The second thesis can be derived from the first by contraposition, DN, and Eq:

$$(p \to \neg q) \to \neg(p \to q)$$

In (Nelson 1936a), he argues for Boethius's theses. He does so by reading $\to$ as 'if ... then'. Nelson objects to the reading of 'if ... then' as material implication. He claims that when one denies that 'if p, then q' she is not asserting p and $\neg q$. When one denies 'if the ground is wet, then it is raining', one does not assert that the ground is wet. She may only be denying the connection between the ground's being wet and rain. Nelson asserts that the contradictory of 'if p, then q' is '$p \circ \neg q$' (Nelson 1936a, 507). The contradictory of $p \circ \neg q$ is $\neg(p \circ \neg q)$, i.e., $p \to q$, hence the proper treatment of 'if .. then' in English (at least in some cases) is as $\to$.

Aristotle's first thesis follows straightforwardly from Boethius's first thesis:

1.	$p \to p$	1.1
2.	$(p \to p) \to \neg(p \to \neg p)$	Boethius's first thesis and US
3.	$\neg(p \to \neg p)$	$1, 2,$ MP

Aristotle's second thesis follows in the same way, but with the use of $\neg p \to \neg p$ as a substitution instance of 1.1 and substituting $\neg p$ for p in Boethius's first thesis.

NL altered by the removal of the prefix $(p \neq q \neq r)$ from axiom 1.5 to produce a standard form of conjunctive syllogism $(((p \to q) \wedge (q \to r)) \to (p \to r))$, yields a subsystem of Richard Routley and Hugh Montgomery's connexive logic Z1c (Routley and Montgomery 1968). Let us call this logic with the removal of the prefix from 1.5, NL^-. This logic is very closely related to weak relevant logics. If we define $A \circ B$ as $\neg(A \to \neg B)$ and $A|B$ as $A \to \neg B$, then all the axioms of NL^-, except 1.4, are provable in the relevant logic DK. Whether NL (or even NL^-) should be considered a relevant logic is a technical issue that we set aside here.

6 Contemporary connexive approaches

Soon after Nelson finished his dissertation and published (Nelson 1930), some other philosophers adopted similar views.

According to (Baylis 1931), it is true that p implies q iff p is inconsistent with $\neg q$, but this is rather a consequence of the definition of implication rather than the definition itself. We have that p implies q (or else, is *subsumed* under q) in case the intensional meaning of q is identical with a part of the intensional meaning of p.

(Wisdom 1934) criticises Lewis' definition of consistency, taking sides with Nelson also on the issue of paradoxes of strict implication.

Duncan-Jones claims to have arrived independently at an analysis of paradoxes similar to Nelson's:

> The general point of view which I am adopting is much the same as that of Everett J. Nelson in "Intensional relations", *Mind* 1930. I had already worked out this paper in outline before I discovered the existence of Nelson's paper: otherwise the note would have been based directly on Nelson. I differ from Nelson on one or two points to be mentioned below. (Duncan-Jones 1935, 70)

Such points are actually not quite minor. Duncan-Jones expresses doubts about Nelson's intensional understanding of conjunction and about his ensuing rejection of conjunctive simplification. To rescue this principle, Duncan-Jones distinguishes between implying and *invoking*. We have that p invokes q if p entails q and p is not a conjunction which would still entail q if one of its conjuncts were dropped. The paradoxes of strict implication, thus, are fallacies of equivocation: conjunctive simplification holds for implication but not for invocation, while antilogism holds for the latter but not for the former.

7 Lewis's reply to Nelson

Lewis does not reply to Nelson by name. In fact, Lewis rarely publishes explicit replies to criticisms of either his logical or philosophical views. But we do find in his work, some discussion and a clear rejection of connexive logic.

Lewis briefly discusses Nelson's axiom 1.4 in *Symbolic Logic* (Lewis and Langford 1951, 157):

> The principle $(p \prec q) \prec (p \circ q)$, which might be expected to hold, does not, in fact, hold without exceptions. There are propositions which are not consistent with any other propositions; yet such

> propositions have implications. Hence if $p \multimap q$ holds, it requires the further condition $p \circ p$ to assure that p is consistent with q.

He also argues against Aristotle's thesis. He points out that, in his system, whenever $p \multimap q$ is a theorem, $p \wedge \neg q$ entails its own negation.

In an appendix to the second edition of *Symbolic Logic* written in January 1959, he acknowledges that "on account of the paradoxes there are many who doubt that" S2 is the logic of entailment, that it captures the intuitive notion of deducibility. He points out that there are alternative logics that avoid the paradoxes but that these all fail to contain certain "indispensable rules of inference" (Lewis and Langford 1959, 511–512).

As we have said, Lewis does not mention Nelson by name, but he attacks Nelson's rejection of the simplification thesis:

> If the moon is a planet and not made of green cheese, then the moon is a planet. But if two premises together imply a conclusion and the conclusion is false while one of the premises is true, then the other premise must be false. So if the moon is a planet and also is not a planet, then the moon is made of green cheese. We need not dally with a suppositious amendment: "Whatever in the premises is non-essential to the conclusion is no part of what implies the conclusion." This dictum would condemn the syllogism, since in every syllogism the premises contain information not contained in the conclusion. (Lewis and Langford 1959, 512)[12]

In this passage, Lewis is attributing to Nelson the requirement that whatever information is in the premises must appear in the conclusion. We doubt that Nelson in fact held this. But if he did, surely the only general principle of logic that he could accept is the thesis of identity, i.e., $p \to p$, and Lewis says that this is all that can be justified if we accept the "suppositious amendment" (ibid.). We interpret Nelson as holding a strong form of real use requirement, that all the premises need to be used to derive a conclusion, not that all the information in the premises survives in the conclusion.

At any rate, Lewis clearly rejects connexive logic because he thinks that simplification is an indispensable rule of inference, and that the corresponding

[12]Notice the use of antilogism in this passage. The argument is in fact a truncated version of Lewis's early independent argument discussed in section 2.

thesis, $(p \wedge q) \rightarrow p$, is required by a logic that is meant to capture the intuitive notion of deducibility.

8 Lewis's and Nelson's later philosophies of logic

Although, in his later work, Lewis rejects the core ideas behind connexive logic, there are some similarities in their philosophies of logic in their later work.

Nelson writes little formal logic after the mid-1930s, but he does write papers on the philosophy of logic in that period. In those paper, he retains his Platonism. In a paper in 1949 he says:

> I am convinced that no [...] analysis or criticism can be supported except by employing assumptions rooted in speculative philosophy. Thus in the purely abstract fields such as logic and mathematics we run into the problem of universals .... (Nelson 1949, 610)

In this passage Nelson is stating that intuitions about logical truth has to do with perceived or understood connections between abstract entities (universals). Nelson thinks that "acquaintance with instances of qualities or relations reveals that the essence of qualities and relations involves conformity with logical laws" (ibid. pp 612–613). The use of the word "acquaintance" would indicate to Nelson's readership that he has a position similar to Russell's theory of acquaintance with universals. At any rate, Nelson holds some version of platonism in his thesis and still maintains this view twenty years later.

Lewis did not change the formal logic that he accepted after 1930. His preferred logic remains S2 until the end of his life. But Lewis does change his philosophy of logic to look a lot like Nelson's. In a 1935 manuscript, Lewis says:

> Let us begin with the interpretation of $p \prec q$, which is intended to be synonymous with q is deducible from p. We conceive that all valid deduction takes place through the analysis of meanings or connotations. Observation of the truth or falsity of propositions

> may, in specific instances, show that one is not deducible from another - a false proposition is never deducible from a true one – but obviously such examination of truth-values, without analysis of propositions, can never demonstrate positively that q is deducible from p. Socrates is a man implies Socrates is an animal because the connotation of man includes the connotation of animal. Connotations are (in the language of the older logic) universals; and any relation of connotations as such is a universal relation. If q is deducible from p, this is because analysis of the meanings of the propositions p and q will reveal a universal relation of connotation or intension (Lewis and Langford 2014, 46)[13]

Lewis continues to develop this sort of semantic realism in his published work. A decade or so later in (Lewis 1946), he develops a more complex theory of abstract objects as the metaphysical basis of his theory of meaning. In (Lewis 1951), he treats statements again as expressing universals, that is as properties of worlds. Thus, in Lewis's later writings he adopts a metaphysical realist philosophy of logic similar to Nelson's.

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[13] Although Langford is listed as a co-author of this paper, both he and Lewis claimed that Langford had very little input into it.

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RESEARCH ARTICLE

Alternative Axiomatizations of the Conditional System VC

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Abstract: The central result of the paper is an alternative axiomatization of the conditional system **VC** which does not make use of Conditional Modus Ponens: $(A > B) \supset (A \supset B)$ and of the axiom-schema CS: $(A \wedge B) \supset (A > B)$. Essential use is made of two schemata, i.e. X1: $(A \wedge \Diamond A) \supset (\Diamond A >< A)$ and T: $\Box A \supset A$, which are subjoined to a basic principle named Int: $(A \wedge B) \supset (\Diamond A > \Diamond B)$. A hierarchy of extensions of the basic system **V** called **VInt**, **VInt1**, **VInt1T** is then construed and submitted to a semantic analysis. In Section 3 **VInt1T** is shown to be deductively equivalent to **VC**. Section 4 shows that in **VC** the thesis X1 is equivalent to X1∨: $(\Diamond A >< A) \vee (\Diamond \neg A >< \neg A)$, so that **VC** is also equivalent to a variant of **VInt1T** here called **VInt1T**^{o}. In Section 6 both X1 and X1∨ offer the basis for a discussion on systems containing CS, in which it is argued that they cannot avoid various kinds of partial or full trivialization of some non truth-functional operators.

Keywords: Conditional logic; centering condition; trivialization; modal collapse.

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1 The conditional systems V, VW, VC and C2

The system of conditional logic named **VC** in (Lewis 1973) (but named **C1** in Lewis 1971) is currently considered the central system inside the family of so-called "classical conditional logics". (Lewis 1973, 130) writes: "In my analysis of counterfactuals, I officially imposed centering and none of the other conditions, **VC** is therefore my official logic for the counterfactual interpretation". However, Lewis was not fully satisfied with the various axiomatizations he found for this system: for this reason he was inclined to prefer an axiomatization whose primitive notion was that of comparative possibility, here represented by the symbol $\prec$.

The axiomatization which we will use as a starting point may be found in (Nute 1980). In what follows A, B, C,... will be (meta)variables for wffs, $\neg$, $\supset$, $\wedge$, $\vee$ and $\equiv$ will be symbols for the standard truth-functional operators, while the symbol $>$ will be used for the primitive conditional operator, $><$ for conditional equivalence and $\Supset$ for the dual of $>$. The relation symbolized by $A \Supset B$ will be read as "A is cotenable with B". The formation rules are standard. Parentheses will be omitted when no ambiguity arises around wffs having $>$, $\Supset$ or $><$ as the main operator.

The definitions of the auxiliary operators are:

Def $><$	$A >< B =_{df} A > B \wedge B > A$;
Def $\Supset$	$A \Supset B =_{df} \neg(A > \neg B)$;
Def $\Box$	$\Box A =_{df} \neg A > A$;
Def $\Diamond$	$\Diamond A =_{df} \neg(A > \neg A)$;
Def $\dashv$	$A \dashv B =_{df} \Box(A \supset B)$.

The relation of comparative possibility $A \prec B$ (which will not be used in the present paper) is defined by Lewis as $\Diamond A \wedge ((A \vee B) > (A \wedge \neg B))$, while $A \preccurlyeq B$ stands for $\neg(B \prec A)$.

The axiom schemata of **VC**, according to (Nute 1980, 129), may be formulated as follows:

PC	All the tautologies of the truth-functional propositional calculus;
ID	$A > A$;
MOD	$\neg A > A \supset B > A$;

CSO $A >< B \supset (A > C \equiv B > C)$;
CV $(A > B \wedge A \Supset C) \supset (A \wedge C) > B$;
CMP $A > B \supset (A \supset B)$;
CS $(A \wedge B) \supset A > B$.

The rules are:

MP From $\vdash A$ and $\vdash A \supset B$ to infer $\vdash B$;
RCEC From $\vdash A \equiv B$ to infer $\vdash C > A \equiv C > B$;
RCK From $\vdash (A_1 \wedge ... \wedge A_n) \supset B$
to infer $\vdash (C > A_1 \wedge ... \wedge C > A_n) \supset C > B$, for $n \geq 0$.

The following rule RCE is derivable as a special case of RCK:

RCE From $\vdash A \supset B$ to infer $\vdash A > B$.

The axiom schema CSO expresses a particular feature of any conditional logic which contains it: the possibility of replacing $><$-equivalents in the antecedent of a conditional. The rule

RCEA From $\vdash A \equiv B$ to infer $\vdash A > C \equiv B > C$

follows from CSO and RCE, since RCE yields the derived rule

RCEq From $\vdash A \equiv B$ to infer $\vdash A >< B$.

It follows that replacement of proved $\equiv$-equivalents is a universally valid rule in **VC**. In what follows we shall call *a*-equivalents any two statements which can be reciprocally replaced in the antecedent of a conditional via an application of CSO or of RCEq. The subsystem of **VC** which results from removing axiom CS is called **VW**. Removing CMP (the so-called *Conditional Modus Ponens*) from **VW** the resulting minimal system is called **V**. Any conditional system which has among its rules RCEC and RCEA is called *classical* in (Chellas 1975), so **V** and all its extensions are classical conditional logics.

Lewis showed that the fragment of **VW** and **VC** containing only $\Box$ and truth-functional operators is exactly **KT**, while the modal fragment of **V** is the minimal system **K**. The collapse-formula (Ban) $A \supset \Box A$ is underivable in all extensions of **VC** written in the $>$-language considered in (Lewis 1973), so none of these systems is trivial.[1] If $\top$ stands for $A \vee \neg A$ and $\bot$ for $\neg\top$, the collapse is however derivable in **VC** for a second modal operator $\boxdot$ defined by

Def $\boxdot$ $\boxdot A =_{df} \top > A$.

The collapse of $\boxdot$ is a consequence of axiom CS. In fact, putting $\top$ in place of A in CS the result is $(\top \wedge B) \supset \top > B$, so $B \supset \boxdot B$. This formula will be called Ban$\boxdot$. The stronger equivalence $B \equiv \boxdot B$ will be called Triv$\boxdot$ and is also derivable in **VC**, given that $\top > B$ (i.e. $\boxdot B$) implies $\top \supset B$ by CMP, hence B by **PC**.

Various versions of the model theory for conditional language have been proposed in the literature (see for instance Lewis 1971). For our aims the semantics for conditional systems may be outlined as follows. A **V**-model is a 4-ple $\langle W, f, R, V\rangle$, where:

(i) W is a non-empty set of possible worlds $\{a, b, c, ...\}$,

(ii) f is a function from couples of propositions and worlds to sets of worlds such that, for every proposition A and world i, $f(A, i)$ is a subset of W which is also a subset of A-worlds, i.e. of worlds where A is true (intuitively, the set of A-worlds "more similar" to i). The properties of f are the basic properties of so-called selection functions, as formulated by (Stalnaker 1968), with the essential difference that the selection function f here has sets of worlds as its values (*set-selection functions*) while Stalnaker asks that f selects a unique world.[2]

(iii) R is a binary relation on W;

(iv) if j belongs to $f(A, i)$ then iRj;

(v) V is an evaluation function that is defined in a standard way as far as

[1] In (Lewis 1973, 121), a trivial system named **CA** is formulated in a language containing the symbol $\prec$. The trivializing schema is $A \prec B \equiv A \supset B$.

[2] (Lewis 1971, 75) proposes the following minimal clauses for the definition of (non-centered) set-selection functions, where A is a sentence and $[A]$ the set of A-worlds:

(a) $f(A, i) \subseteq [A]$ (which means that $V(A, j) = 1$ at every j in $f(A, i)$);

(b) if $f(A, i) \subseteq [B]$ and $f(B, i) \subseteq [A]$, then $f(A, i) = f(B, i)$;

(c) either $f(A \vee B, i) \subseteq [A]$ or $f(A \vee B, i) \subseteq [B]$ or $f(A \vee B, i) = f(A, i) \cup f(B, i)$.

We are assuming in what follows that the function f has the properties stated in (a), (b), (c).

truth-functional operators are concerned and that w.r.t. the >-operator satisfies the following clause:

(vi) $V(A > B, i) = 1$ iff $V(B, j) = 1$ at every j in $f(A, i)$ (this means that if $f(A, i)$ is $\emptyset$ the right part of the equivalence (vi) is vacuously satisfied: so, in this special case, $V(A > B, i) = 1$ for every B, so $V(A > \neg A, i) = V(\Box\neg A, i) = 1$. On the other hand, if $\Box\neg A$ is true at i, $f(A, i)$ is $\emptyset$; otherwise, if $f(A, i) \neq \emptyset$, by clause (iv) some j exists such that iRj, so $\Diamond A$ is true at i);

A **VW**-model has all the properties (i)-(vi) and furthermore the following:

- (Weak Centering) if $V(A, i) = 1$ then $i \in f(A, i)$.

Weak Centering validates CMP, so its instantiation $\neg A > A \supset (\neg A \supset A)$, so $\Box A \supset A$. Then in **VW**-models the relation R has the property of being reflexive.

A **VC**-model has all the properties of **VW**-models and also the following:

- (Centering) If $V(A, i) = 1$, then $f(A, i) = \{i\}$

which validates axiom CS. Stalnaker's system **C2** has in place of CS the so called "Conditional Excluded Middle" (CEM):

CEM $A > B \vee A > \neg B$.

C2-models are as **VC**-models with the only difference that the set $f(A, i)$ is a singleton. As proved by (Lewis 1973), **V**, **VW**, **VC**, **C2** are sound and complete for the above defined semantics.

2 The conditional systems VInt, VInt1 and VInt1T

We remark that all the schemata used in the above introduced axiomatization of **VC**, with the only exception of Id, are schemata involving two metavariables. In what follows we will consider three schemata, two of which contain a single metavariable, which can be used to provide an alternative axiomatization of **VC**:

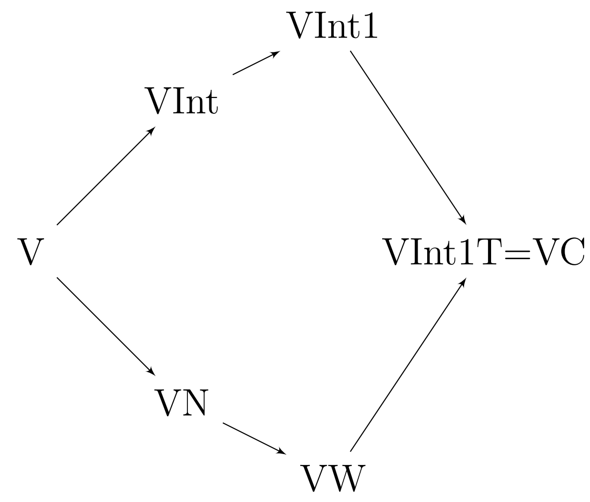

Figure 1: Converging extensions of **V**

Int $(A \wedge B) \supset \Diamond A > \Diamond B$;
X1 $(A \wedge \Diamond A) \supset \Diamond A >< A$;
T $\Box A \supset A$.

A wff which is equivalent to Int is the schema:

Int+ $(A \wedge B) \supset \Diamond A >< \Diamond B$.

The diagram in Figure 1 offers a visualization of the inclusion relation (represented by the arrow) among six conditional systems with the following names:

VInt = **V** + Int;
VInt1 = **V** + Int + X1;
VInt1T = **V** + Int + X1 + T;
VN = **V** + N (N : $\Diamond\top$);[3]
VW = **V** + CMP;
VC = **V** + CMP + CS.

The right side of the diagram in Figure 1 is a fragment of the map of conditional logics reproduced in Fig. 5 of (Lewis 1973, 131). Other combinations of schemata (for instance **V** + X1) will not be taken here in consideration for sake of simplicity. In what follows we prove that the systems in the lower side of the diagram are distinct systems, or more exactly that each one is properly included in the next of the sequence. In other words we have to prove that Int is not a **V**-theorem, X1 is not a **VInt**-theorem and that T is not a **VInt1**-theorem. The first result is simple since we have at our disposal a soundness and completeness theorem for **V** w.r.t. the class of **V**-models.

Proposition 2.1. *Int is not a* ***V****-theorem.*

Proof. It is enough to prove that Int, i.e. $(A \wedge B) \supset \Diamond A > \Diamond B$, is falsified in a **V**-model with the following consistent properties: $W = \{i, j\}$ and for any formulas A and B, $j \in f(\Diamond A, i)$, $V(A, i) = V(B, i) = 1$, $V(B, j) = 0$, jRj. In this model then $V(\Diamond B, j) = 0$. Since $V(\Diamond A, j) = 1$ for the properties of f (see footnote 2, clause a) $V(\Diamond A > \Diamond B, i) = 0$. Given that $V(A \wedge B, i) = 1$, Int has value 0 at i. Since all **V**-theorems are valid in all **V**-models for the soundness theorem, this amounts to a disproof of Int in **V**. □

In order to reach the other expected results we associate a semantics to the relevant systems in this way. First we define two properties of **V**-models which we call *f-Symmetry* and *Pseudo-centering*.

(*fS*) For every i in W and for every A such that $V(A, i) = 1$, if $j \in f(\Diamond A, i)$, then $i \in f(\Diamond A, j)$.

(PsC) For every i in W, and for every A such that $V(A, i) = 1$ then, for every j in $f(A, i)$, $f(A, j) = \{j\}$.

A consequence of (*fS*) is:

(fS^o) For every i in W and for every A such that $V(A, i) = 1$, if $j \in f(\Diamond A, i)$, then iRj and jRi.

A consequence of (PsC) is:

(PsC^o) For every i in W, and for every A such that $V(A, i) = 1$ then, for every j in $f(A, i)$, jRj.

We prove:

Proposition 2.2. ***VInt*** *is sound w.r.t. the class of f-symmetrical* ***V****-models.*

Proof. Suppose that Int has value 0 at some i of some **V**-model M which is also f-symmetrical. Then $V(A, i) = V(B, i) = 1$ and $V(\Diamond A > \Diamond B, i) = 0$.

This means that there is a j belonging to $f(\Diamond A, i)$ such that $V(\Diamond A, j) = 1$ and $V(\Diamond B, j) = 0$. But due to the property of f-symmetry, $i \in f(\Diamond A, j)$. By the properties of **V**-models, (fS^o) implies that jRi. So, given that $V(\Diamond B, j) = 0$, B has value 0 at all worlds k such that jRk: thus, given that jRi, $V(B, i) = 0$ at i, contrary to the supposition that $V(B, i) = 1$.

One may check by induction on the length of proofs that all **VInt**-axioms schemata are valid for this semantics and that the rules preserve validity, so every **VInt**-thesis is valid in all **V**-models which are f-symmetrical. □

Proposition 2.3. ***VInt1*** *is sound w.r.t. the class of all f-symmetrical and pseudo-centered* ***V****- models.*

Proof. In Proposition 2.2 it is proved that Int is valid in the class of f-symmetrical **V**-models, so it is *a fortiori* valid in the class of pseudo-centered and f-symmetrical models. Since X1 is $(A \wedge \Diamond A) \supset \Diamond A >< A$, it is equivalent to $((A \wedge \Diamond A) \supset (A > \Diamond A)) \wedge ((A \wedge \Diamond A) \supset (\Diamond A > A))$.

First suppose that $(A \wedge \Diamond A) \supset A > \Diamond A$ has value 0 at some world i. Then $V(A, i) = 1$ and $V(A > \Diamond A, i) = 0$. So $V(A, j) = 1$ and $V(\Diamond A, j) = 0$ at some j such that $j \in f(A, i)$. By Pseudo-centering, since $V(A, i) = 1$, this implies $f(A, j) = \{j\}$. But this implies that jRj and $V(A, j) = 1$. It follows that $\Diamond A$ is true at j: contradiction. Secondly we consider $(A \wedge \Diamond A) \supset \Diamond A > A$. Suppose there is an i such that $A \wedge \Diamond A$ has value 1 in i and $\Diamond A > A$ value 0 at i. This means that there is a j such that $j \in f(\Diamond A, i)$, $\Diamond A$ has value 1 and A value 0 at j. By Pseudo-centering, given that $V(\Diamond A, i) = 1$ and $j \in f(\Diamond A, i)$, $f(\Diamond A, j) = \{j\}$. Now f-symmetry states that, for every $i \in W$ and for every A s.t. $V(A, i) = 1$, if $j \in f(\Diamond A, i)$, then $i \in f(\Diamond A, j)$. But $f(\Diamond A, j)$ is $\{j\}$. Thus i belongs to $\{j\}$, which means that $i = j$. Since A has value 0 at j, this means that at i A has value 1 by hypothesis and also value 0, which is impossible. So both Int and X1 are valid in all pseudo-centered and f-symmetrical **V**-models. The expected result follows from standard induction on the length of proofs. □

The proof that the systems **VInt** and **VInt1** are distinct is as follows.

Proposition 2.4. ***VInt1*** *and* ***VInt*** *are non-equivalent systems.*

Proof. It is enough to show that X1 is not a **VInt**-thesis. Let M be a **V**-model with the following consistent properties. $W = \{i, j\}$; $V(A, i) = 1$; $V(A, j) = 0$; iRi; $j \in f(\Diamond A, i)$ and $i \in f(\Diamond A, j)$. It is straightforward to see that M is f-symmetrical, so it is a **VInt**-model. Furthermore $V(\Diamond A, j) = 1$ by the properties of f. Given that $V(\Diamond A, j) = 1$ and $V(A, j) = 0$, so $V(\Diamond A > A, i) = 0$. Also, given that iRi and $V(A, i) = 1$, $V(\Diamond A, i) = 1$. So $V((A \wedge \Diamond A) \supset ((\Diamond A > A), i) = 0$ in M and $V((A \wedge \Diamond A) \supset (\Diamond A >< A), i) = 0$ in M. Since all **VInt**-theses are valid in all f-symmetrical models, X1 is not a **VInt**-thesis. □

The next result concerns the distinction of **VInt1** and **VInt1T**.

Proposition 2.5. ***VInt1*** *and* ***VInt1T*** *are non-equivalent systems.*

Proof. It is enough to show that the axiom-schema T is underivable in **VInt1**. Let us consider a model M with the following properties: $W = \{i\}$, $f(A, i) = \emptyset$, $V(A, i) = 1$. Since $f(A, i) = \emptyset$, due to the properties of selection functions, no j exists such that iRj, so $\neg\Diamond A$ has value 1 at i. Consequently $f(\Diamond A, i)$ is also $\emptyset$: otherwise $\Diamond A$ would have value 1 at some j s.t. iRj, so at i (given that $W = \{i\}$), but this contradicts $V(\neg\Diamond A, i) = 1$. Since $f(\Diamond A, i) = f(A, i) = \emptyset$, the conditions "if $k \in f(\Diamond A, i)$, then $i \in f(\Diamond A, k)$" and "for every $j \in f(A, i)$, $f(A, j) = \{j\}$", jointly required for **VInt1**-models, are vacuously satisfied. Thus M is a **V**-model which is f-symmetric and pseudo-centered, so it is a **VInt1**-model. Since A and $\neg\Diamond A$ have both value 1 at i, $V(A \supset \Diamond A, i) = 0$. If T were a **VInt1**-thesis, $A \supset \Diamond A$ would be also such, which is impossible. □

VInt1 does not even have as a subsystem any system which contains the deontic axiom-schema N: $\Diamond\top$.

Proposition 2.6. $\Diamond\top$ *is not a* ***VInt1****-thesis.*

Proof. The same model M built in Proposition 2.5 may be used to prove that $\Box A$ and $\neg\Diamond A$ have both value 1 in M, so the formula $\Box A \supset \Diamond A$ is falsified in M. Since $\Box A \supset \Diamond A$ is **V**-equivalent to $\Diamond\top$, $\Diamond\top$ cannot be a **VInt1**-thesis. □

The relation between the systems represented on the upper side of the diagram of Figure 1 and the ones on the right side are as follows. We call *incomparable* two systems X and Y when X is not contained in Y and Y is not contained in X.

Proposition 2.7. ***VN*** *is incomparable with* ***VInt****.*

Proof. Since N is $\Diamond\top$, it cannot be derived in **VInt**, otherwise it would derivable in **VInt1**, which is excluded by Proposition 2.6. On the other hand, **VInt** is not a subsystem of **VN**. By Proposition 2.1 Int is not a **V**-theorem and the model M built in the proof of Proposition 2.1 is a serial one, where seriality is the property by which, for every world m in M there is a world n such that mRn. By a well-known result, N is valid in all serial models, so it is valid in M. So Int is falsified in a model which satisfies all **VN**-theorems, so it is not a **VN**-thesis. □

Proposition 2.8. ***VW*** *is incomparable with* ***VInt1****.*

Proof. From the axiom CMP, i.e. $A > B \supset (A \supset B)$, it follows by instantiation $\neg A > A \supset (\neg A \supset A)$, so $\Box A \supset A$, i.e. the schema called T. But we already proved that T is underivable in **VInt1**, so **VW** is not a subsystem of **VInt1**. On the other hand Int, i.e., $(A \wedge B) \supset \Diamond A > \Diamond B$, is underivable in **VW**. A **VW**-countermodel M to Int may be built as follows. $W = \{i, j, k\}$, $V(A, i) = V(A, k) = V(B, i) = 1$ and $V(B, j) = V(B, k) = 0$; $j \in f(\Diamond A, i)$; jRk. Furthermore $i \in f(A, i)$, $i \in f(B, i)$, $j \in f(\neg B, j)$, $k \in f(A, k)$, $k \in f(\neg B, k)$. From this definition it follows that M is a **VW**-model since it has the property of Weak-Centering (i.e. that, for every A and every h, if $V(A, h) = 1$, $h \in f(A, h)$). It also follows from the definition that iRi, jRj, kRk, i.e. total reflexivity. Since jRk and $V(A, k) = 1$, $V(\Diamond A, j) = 1$. But $V(\Diamond B, j) = 0$ since jRk, jRj and B has value 0 in both k and j. Given that $j \in f(\Diamond A, i)$, it turns out that $A \wedge B$ has value 1 while $\Diamond A > \Diamond B$ has value 0 at i. So M is a **VW**-countermodel for Int and Int cannot be a **VW**-theorem. □

3 The deductive equivalence between VInt1T and VC

We now may prove the deductive equivalence between **VInt1T** and **VC**. For the desired result we need the following two Lemmas.

Lemma 3.1. *RB:* $\Diamond A \supset (A > B \supset A \Supset B)$ *(Restricted Boethius) and PCI:* $\Box\neg A \supset A > B$ *(Paradox of Conditional Implication) are* ***V****-theorems.*

Proof. Assume by Reductio the negation of (RB), i.e. $\Diamond A \wedge A > B \wedge A > \neg B$. This implies, in **V**, $\Diamond A \wedge (A > (B \wedge \neg B))$, $\Diamond A \wedge (A > \bot)$, so $\Diamond A \wedge \neg\Diamond A$: contradiction. Then RB is a **V**-theorem.

(PCI) By rule RCK, $\vdash_{\mathbf{PC}} \neg A \supset (A \supset \bot)$ implies $\vdash A > \neg A \supset A > (A \supset \bot)$. So again by RCK, by Id and by Transitivity of $\supset$, $\vdash A > \neg A \supset A > \bot$. $\vdash_{\mathbf{PC}} \bot \supset B$ implies, again by RCK, $\vdash A > \bot \supset A > B$. So, by Transitivity of $\supset$ and by Def $\Box$, $\Box\neg A \supset A > B$. $\Box$

Proposition 3.2. *In* ***VInt1T*** *X1 is equivalent to X1*^{o}*:* $A \supset \Diamond A >< A$.

Proof. A theorem of **VInt1T** is $A \supset \Diamond A$, so $(A \wedge \Diamond A) \equiv A$. Then X1, i.e. $(A \wedge \Diamond A) \supset (\Diamond A >< A)$, by Replacement of Material Equivalents turns out to be equivalent to $A \supset \Diamond A >< A$. $\Box$

Proposition 3.3. *For any formula A, A is a* ***VInt1T****-thesis iff A is* ***VC****-thesis.*

Proof. ($\Longrightarrow$) All the three wffs which are axioms schemata of **VInt1T** are theorems of **VC**. In fact, (a) T: $\Box A \supset A$ and its variant $A \supset \Diamond A$ follow from CMP. (b) An instantiation of CS is $(\Diamond A \wedge \Diamond B) \supset \Diamond A > \Diamond B$ but, given the KT-theorem $(A \wedge B) \supset (\Diamond A \wedge \Diamond B)$, by transitivity of $\supset$ we reach $(A \wedge B) \supset \Diamond A > \Diamond B$, i.e. Int. (c) An instance of CS is $(A \wedge \Diamond A) \supset \Diamond A > A$. Since $\vdash A > \Diamond A$ follows from $\vdash A \supset \Diamond A$ by RCE, it is straightforward to derive $(A \wedge \Diamond A) \supset \Diamond A >< A$, i.e. X1.

($\Longleftarrow$) The following is a syntactical proof of CS in **VInt1T**.

1.	$A \supset \Diamond A >< A$	X1^o
2.	$B \supset \Diamond B >< B$	X1^o
3.	$(A \wedge B) \supset \Diamond A >< A$	(1), $\vdash (A \wedge B) \supset A$, **PC**
4.	$(A \wedge B) \supset \Diamond B > B$	(2), **PC**
5.	$A > B \supset (A \Supset C \supset (A \wedge C) > B)$	CV, **PC**
6.	$\Diamond B > B \supset (\Diamond B \Supset \Diamond A \supset (\Diamond B \wedge \Diamond A) > B)$	(5), $\Diamond B$ for A, $\Diamond A$ for C
7.	$(A \wedge B) \supset (\Diamond B \Supset \Diamond A \supset (\Diamond A \wedge \Diamond B) > B)$	(4), (6), **PC**
8.	$\Diamond A \supset (A >< B \supset A \Supset B)$	RB, **PC**
9.	$\Diamond\Diamond A \supset (\Diamond A >< \Diamond B \supset \Diamond A \Supset \Diamond B)$	(8), $\Diamond A$ for A, $\Diamond B$ for B
10.	$(A \wedge B) \supset \Diamond A >< (\Diamond A \wedge \Diamond B)$	Int, $\vdash_{\mathbf{V}} (\Diamond A >< (\Diamond A \wedge \Diamond B)) \equiv \Diamond A >< \Diamond B$
11.	$\Diamond\Diamond A \supset ((A \wedge B) \supset \Diamond A \Supset \Diamond B)$	(9), Int, **PC**
12.	$\Diamond\Diamond B \supset ((A \wedge B) \supset \Diamond B \Supset \Diamond A)$	(11), B for A and A for B, **PC**
13.	$\Diamond\Diamond B \supset ((A \wedge B) \supset (\Diamond A \wedge \Diamond B) > B)$	(12), (7), **PC**
14.	$\Diamond\Diamond B \supset ((A \wedge B) \supset \Diamond A > B)$	(13), (10), CSO, **PC**
15.	$\Diamond\Diamond B \supset ((A \wedge B) \supset A > B)$	(14), (3), CSO
16.	$\neg\Diamond\Diamond B \supset ((A \wedge B) \supset C)$	$\vdash_{\mathbf{KT}} \Box\Box\neg B \supset \neg B$, $\vdash_{\mathbf{PC}} \neg B \supset ((A \wedge B) \supset C)$
17.	$\neg\Diamond\Diamond B \supset ((A \wedge B) \supset A > B)$	(16), $A > B$ for C
18.	$(A \wedge B) \supset A > B$	(17), (15), **PC**

It remains to be proved that CMP, i.e. $A > B \supset (A \supset B)$, is a theorem of **VInt1T**. From RB: $\Diamond A \supset (A > B \supset A \Supset B)$ (See Lemma 3.1) and CS: $(A \wedge B) \supset A > B$ one obtains $\Diamond A \supset ((A \wedge B) \supset A \Supset B)$, so $\Diamond A \supset (A > \neg B \supset (A \supset \neg B))$ and obviously

(*) $\Diamond A \supset (A > B \supset (A \supset B))$.

But $\neg\Diamond A \supset (\neg A \vee B)$ is a **KT**-thesis, so *a fortiori*:

(**) $\neg\Diamond A \supset (A > B \supset (A \supset B))$

is such. Then by (**) and (*) $A > B \supset (A \supset B)$, i.e. CMP, follows in a straightforward way.

□

As proved by Lewis, **VC** is sound and complete w.r.t. the class of all centered models. A simple semantic remark helps to understand the equivalence between the two systems **VC** and **VInt1T**.

Proposition 3.4. *Any **V**-model M is centered if and only if M is pseudocentered, reflexive and f-symmetric.*

Proof. (a) It is easy to see that every centered **V**-model M is *a fortiori* pseudocentered, reflexive and f-symmetric. Reflexivity follows obviously by Centering. Furthermore, if $V(A, i) = 1$, by Reflexivity $V(\Diamond A, i) = 1$, so by Centering $f(\Diamond A, i) = \{i\}$: hence $f(A, i) = f(\Diamond A, i)$, given that both are identical to $\{i\}$. Hence if $j \in f(A, i)$, $j = i$.

The two properties *fS* and *PsC* follow in the following way.

(*fS*) For every i in W such that $V(A, i) = 1$, if $j \in f(\Diamond A, i)$, by the proved identity $f(A, i) = f(\Diamond A, i)$, $j \in f(A, i)$; then by the identity $j = i$, $i \in f(\Diamond A, j)$. So, supposing that $V(A, i) = 1$, if $j \in f(\Diamond A, i)$, $i \in f(\Diamond A, j)$.

(*PsC*) For every i in W, if $V(A, i) = 1$ then, being proved by the Centering property that $f(A, i) = \{i\}$, it follows that $i = j$ for every j s.t. $j \in f(A, i)$. So, if $V(A, i) = 1$, by substitution of j to i $f(A, j) = \{j\}$, which is the property of Pseudo-centering.

b) It may be proved that every pseudo-centered f-symmetric reflexive model is a centered model. Suppose $V(A, i) = 1$. Then by Reflexivity $V(\Diamond A, i) = 1$. One can prove that $f(\Diamond A, i)$ is not $\emptyset$: $f(\Diamond A, i) = \emptyset$ in fact would imply the truth at i of $\Diamond A > C$ for every C, so the truth of $\Diamond A > \neg\Diamond A$, of $\Box\neg\Diamond A$ (by Def $>$) and of $\Box\Box\neg A$, so by reflexivity the truth of $\neg A$ at i, which is the negation of the hypothesis $V(A,i) = 1$. Then there is a $j \in f(\Diamond A, i)$, and in such j $\Diamond A$ has value 1. Now Pseudo-centering holds for every wff, so also for $\Diamond A$. Hence, for every i in W, and for every A such that $V(\Diamond A, i) = 1$ we have, for every j in $f(\Diamond A, i)$, $f(\Diamond A, j) = \{j\}$ and we know that there is at least one j of this kind. And, given that $j \in f(\Diamond A, i)$ implies $i \in f(\Diamond A, j)$ by f-symmetry, $i \in \{j\}$ and $j = i$. Since $f(\Diamond A, j) = \{j\}$, by substitution of identicals $f(\Diamond A, i) = \{i\}$. Since A is true at i, it is also true at every world of $f(\Diamond A, i)$, so $\Diamond A > A$ is true at i. Given that $A > \Diamond A$ is true at i and at all the possible worlds of any reflexive model, then $\Diamond A >< A$ has value 1 at i. This means, by property b) of the function f (see section 1), that $f(A, i) = f(\Diamond A, i)$. Given that $f(\Diamond A, i) = \{i\}$, we conclude that, if $V(A, i) = 1$, $f(A, i) = \{i\}$, which expresses the condition of Centering.

$\Box$

4 CS as a source of partial trivialization of modal operators

The two basic laws of central systems of classical conditional logics, CMP and CS, have been object of criticism by various scholars. Conditional Modus Ponens has been criticized by (McGee 1985) and is not a theorem of the system **Å** in (Åqvist 1973). The reason is that if $A > B$ is interpreted by something as $*A \dashv B$ (where $*$ is the so called "circumstantial operator" indicating the conjunction of A with a tacitly understood antecedent), an acceptable version of conditional Modus Ponens would be not $A > B \supset (A \supset B)$ but $A > B \supset (*A \supset B)$: but the latter formula is inexpressable in a language having $>$ as the only primitive. As far as CS is concerned, suffice it to mention (von Kutschera 1974) and (Gundersen 2004) just to quote only two papers in which the conditional systems under analysis omit CS. The principle has been criticized also in (Bennett 1974) and (Bigelow 1976). What we want to stress in this paper is that the implausibility of CS, and more generally of system **VC**, is due to the fact that CS is a source of a partial or full trivialization of some non-truthfunctional operators definable in terms of $>$. As observed at the beginning, if the necessity operator is defined as $\boxdot A =_{df} \top > A$, the collapse-formula Triv$\boxdot$: $\boxdot A \equiv A$ is derivable in **VC** but not in the weaker system, being a consequence of the introduction of CS.

It is of some interest to remark that Def $\boxdot$ and Def $\Box$ allow for a different axiomatization of **VC** consisting in two simple axiom-schemata with only one metavariable. Let us call **V**$\boxdot\Box$ the system which is obtained by extending the axiomatic basis of **V** with the definitions of $\boxdot$ and $\Box$ and the following two schemata:

Ban$\boxdot$	$A \supset \boxdot A$;
T	$\Box A \supset A$.

Then we may prove the following metatheorem.

Proposition 4.1. ***V***$\boxdot\Box$ *and* ***VC*** *are deductively equivalent systems.*

Proof. The rules of the two systems and the axioms of **V** are common to **V**$\boxdot\Box$ and **VC**, so it is enough to do these two steps.

($\Longrightarrow$) We prove that CS and CMP are derivable in **V**$\boxdot\Box$. As for CS:

1. $A \supset \top > A$	Ban$\boxdot$, Def $\boxdot$
2. $A > \top$	**PC**, RCE
3. $A \supset \top >< A$	(1), (2), **PC**
4. $B \supset \top >< B$	(3), B for A
5. $(A \wedge B) \supset (\top >< B \wedge T >< A)$	(3),(4), *Theorema Praeclarum*
6. $\top >< A \supset (\top > B \equiv A > B)$	CSO, $\top$ for A, A for B, B for C
7. $(\top >< A \wedge \top >< B) \supset A > B$	(6), **PC**, $\vdash B > \top$ ((2), B for A)
8. $(A \wedge B) \supset A > B$	(5),(7), **PC**

As for CMP: The same proof as for the wffs (*) and (**) used in the proof of Proposition 3.3, which are steps to derive CMP from T and CS.

($\Longleftarrow$) We have already shown that the schema Ban is derivable in **VC**, while T is a trivial consequence of CMP. $\Box$

Proposition 4.1 implies that Ban$\boxdot$: $A \supset \boxdot A$ and Triv$\boxdot$: $A \equiv \boxdot A$ are equivalent in **V**$\boxdot\Box$, since $\boxdot A \supset A$ easily follows from CMP.

Notice that in systems having $\strictif$ as primitive (see, for instance, **S1**–**S5** introduced in Lewis and Langford 1932) $\Box A$ may be indifferently defined as $\neg A \strictif A$ and $(A \vee \neg A) \strictif A$. However, in conditional logic the distinction between $\boxdot A$ and $\Box A$ depends, via Def $\boxdot$ and Def $\Box$, on the distinction between $(A \vee \neg A) > A$ e $\neg A > A$. The reason why such two formulas cannot be logically equivalent in classical conditional logics is that there is no room in them for the so-called Simplification of Disjunctive Antecedents

SDA $(A \vee B) > C \supset B > C$.

Thanks to SDA, in fact, $(A \vee \neg A) > A$ (i.e., $\boxdot A$) would imply $\neg A > A$ (i.e., $\Box A$), while the converse implication follows from MOD. But the failure of SDA is a weak point of classical conditional logic. (Nute 1980, 40 ff.) stresses with a different argument the incompatibility between SDA and CS, and introduces a class of non-standard conditional logics in which the first, but not the latter theorem, is allowed. As is well-known, Lewis has defended CS by claiming that conditional logic is interested in truth conditions and not in relevance conditions between the clauses. The problem of the "disconnection" of the clauses is traditionally seen as a weak point of the material conditional. This helps to understand a second kind of trivialization yielded by CS in conjunction with CMP: in fact, their combination leads one to prove the equivalence between classical conditionals with a true antecedent and material conditionals with a true antecedent.

A conditional which is stated in conjunction with the truth of its antecedent may be read as a "since" conditional or, as (Goodman 1947) says, a *factual* conditional. Thanks to CMP, the factual conditional $A \wedge A > B$ entails $(A \wedge B) \wedge A > B$ and *vice versa*. Thanks to CS, $A \wedge B$ entails $A > B$, so $(A \wedge B) \wedge A > B$ turns out to be equivalent to $A \wedge B$. So we reach soon the equivalence

($\circ$) $A \wedge B \equiv A \wedge (A > B)$

($\circ$) states that any factual conditional collapses on the conjunction of its clauses. Furthermore, given that $A \wedge B$ is **PC**-equivalent to $A \wedge (A \supset B)$, we have the collapse of factual material conditionals on factual classical conditionals, that is a special case of trivialization

($\circ\circ$) $A \wedge (A \supset B) \equiv A \wedge (A > B)$.

In the present paper attention has been given to a third kind of trivializiation. Let us recall that

X1^o $A \supset \Diamond A >< A$

follows from CS in the system **VC** and that it entails CS in the system **VInt1T**. X1^o asserts that if A is true, A is a-equivalent with its own possibility. Let us read again $A \wedge (A > C)$ as "Since A, C" and let us suppose that $A \wedge (A > C)$ is true (e. g. "Since it rained, streets are wet"). $A \wedge (A > C)$ implies A and, by X1^o, A implies $\Diamond A >< A$. Thanks to CSO, then, from $A > C$ we derive $\Diamond A > C$. And, given that we have at our disposal also the law $A \supset \Diamond A$, we conclude that the conjunction $A \wedge (A > C)$ entails the conjunction $\Diamond A \wedge \Diamond(A > C)$. More formally, we derive the following formula:

$$(\circ \circ \circ) \quad (A \wedge (A > C)) \supset (\Diamond A \wedge (\Diamond A > C)).$$

Following the proposed interpretation of A and C, then the consequent of $(\circ \circ \circ)$ says "streets are wet since it is logically possible that it rained", i.e. is something which is intuitively false, while the antecedent $A \wedge (A > C)$ appears to be intuitively true. One might be willing to remark that this result is simply an apparent anomaly, since the background conception of conditionals which inspired Stalnaker-Lewis systems is divisionist. In other words the background theory claims that there are two kinds of conditionals (roughly: the indicative and the counterfactual) and the truth-conditions for the indicative conditional are the same as the ones for the material conditional. However, the preceding remark should face the content of the following metatheorem:

Proposition 4.2. *In **VC** the thesis X1^o: $A \supset \Diamond A >< A$ is logically equivalent to X1$\vee$: $\Diamond A >< A \vee \Diamond\neg A >< \neg A$.*[4]

Proof. The proof consists in proving the two sides of the equivalence which we call here (ia) and (ib). Let us recall that **VC** contains **VW**, hence the $\Box$-logic **KT** and also the theorems $A \supset \Diamond A$ and $A > \Diamond A$.

[4] $\Diamond A >< A \vee \Diamond\neg A >< \neg A$ should not be confused with $\Box A >< A \vee \Box\neg A >< \neg A$. One may prove that $\Box A >< A \vee \Box\neg A >< \neg A$ is equivalent to $\neg A \supset \Box A >< A$, and the latter formula may be proved to be equivalent to the two-metavariable wff $A \supset (B > (A > B))$, which is not **VC**-valid (the proofs will not be given here).

(ia)

1. $\vdash \Diamond A >< A \vee \Diamond\neg A >< \neg A$	X1$\vee$
2. $(\Diamond A \wedge \neg A) \supset \neg(\Diamond A > A)$	CMP, **PC**, $\Diamond A$ for A, A for B
3. $(\Diamond A \wedge \neg A) \supset \neg(\Diamond A >< A)$	(2), Def $><$, $\vdash A > \Diamond A$
4. $(\Diamond A \wedge \neg A) \supset \Diamond\neg A >< \neg A$	(3), (1), **PC**
5. $\neg\Diamond A \supset \Diamond\neg A >< \neg A$	MOD, $\Diamond\neg A$ for B, $\neg A$ for A, $\vdash \neg A > \Diamond\neg A$
6. $(\neg\Diamond A \wedge \neg A) \supset \Diamond\neg A >< \neg A$	(5), **PC**
7. $\neg A \supset \Diamond\neg A >< \neg A$	$\vdash_{\mathbf{KT}} ((\neg A \wedge \Diamond\neg A) \vee (\neg A \wedge \Diamond A)) \equiv \neg A$, (4), (6), **PC**
8. $A \supset \Diamond A >< A$	(7), $\neg A$ for A, **PC**

(ib)

1. $\vdash A \supset \Diamond A >< A$	X1^o
2. $\neg A \supset \Diamond\neg A >< \neg A$	(1), $\neg A$ for A
3. $((A \supset B) \wedge (C \supset D)) \supset ((A \vee C) \supset (B \vee D))$	**PC**
4. $(A \vee \neg A) \supset (\Diamond A >< A \vee \Diamond\neg A >< \neg A)$	(1), (2), (3) ($\neg A$ for C, $\Diamond A >< A$ for B, $\Diamond\neg A >< \neg A$ for D), MP
5. $\Diamond A >< A \vee \Diamond\neg A >< \neg A$	(4), $\vdash \top$, MP

□

As a consequence of Proposition 4.2, an axiom system which is equivalent to **VInt1T** (so to **VC** and to **V**$\boxdot\Box$) is the following, which we call here **VInt1T**^{o}:

Int	$(A \wedge B) \supset \Diamond A > \Diamond B$;
X1$\vee$	$\Diamond A >< A \vee \Diamond\neg A >< \neg A$;
T	$\Box A \supset A$.

Now the disjunction X1$\vee$ implies that at least one of the two statements A and $\neg A$ is a-equivalent with its own possibility. This statement does not make any assertion containing factual conditionals. Thanks to X1$\vee$ we are able to perform in **VC** the following proof. In order to simplify notation, we recall that $\Diamond\neg A \wedge \Diamond A$ means that A is contingent and is symbolically represented usually by ∇A.

1.	$(A > B \wedge A \Supset C) \supset (A \wedge C) > B$	CV
2.	$\Diamond\neg A > \neg A \supset ((\Diamond\neg A \Supset \Diamond A) \supset (\Diamond\neg A \wedge \Diamond A) > \neg A)$	(1) $\Diamond\neg A$ for A, $\Diamond A$ for C, $\neg A$ for B, **PC**
3.	$\Diamond A >< A \vee ((\Diamond\neg A \Supset \Diamond A) \supset (\Diamond\neg A \wedge \Diamond A) > \neg A))$	X1$\vee$, (2), MP
4.	$((\Diamond\neg A \Supset \Diamond A) \wedge (\Diamond\neg A \wedge \Diamond A) \Supset A) \supset (\Diamond A >< A)$	(3), $\vdash A \vee B \equiv \neg B \supset A$, Def $\Supset$
5.	$((\Diamond\neg A \Supset \Diamond A) \wedge (\nabla A \Supset A)) \supset \Diamond A >< A$	(4), Def ∇
6.	$(\Diamond\neg A \wedge \Diamond A) \supset (\Diamond\neg A \Supset \Diamond A)$	CMP, **PC**
7.	$(\nabla A \wedge (\nabla A \Supset A)) \supset \Diamond A >< A$	(6), (5), **PC**, Def ∇

How can we read (7)? It asserts the if A is contingent and its contingency is cotenable with A, then A and $\Diamond A$ are a-equivalent. But most actual and possible facts of real life have the property stated in the antecedent of this implication. So, given that raining is a contingent fact and that such property does not exclude rain, raining is a-equivalent to its own possibility. Given the mentioned premises, as a result of a-equivalence we may then replace A and $\Diamond A$ in the antecedents of any appropriate conditional. In many cases this replacement however leads to bizarre conclusions. For instance, we can imagine a scenario in which yesterday it did not rain ($\neg A$) but rain was clearly a possibility ($\Diamond A$). Now let us consider the true counterfactual:

(i) if it had rained, streets would have been wet ($A > W$)

and from (i) and the other premises we conclude, exploiting replacement of a-equivalents

(ii) if rain had been a possibility, streets would have been wet ($\Diamond A > W$)

(ii) is not only false but, thanks to the true premise $\Diamond A$, it allows to go further: we may conclude by CMP $\Diamond A \supset W$ and by standard Modus Ponens, from $\Diamond A$, that also W is true, i.e. that streets were wet yesterday: very strange conclusion since among our premises there is the fact that yesterday it did not rain.

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RESEARCH ARTICLE

On a Supposed Puzzle Concerning Modality and Existence

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Abstract: Kit Fine has proposed a new solution to what he calls 'a familiar puzzle' concerning modality and existence. The puzzle concerns the argument from the alleged truths 'It is necessary that Socrates is a man' and 'It is possible that Socrates does not exist' to the apparent falsehood 'It is possible that Socrates is a man and does not exist'. We discuss in detail Fine's setting up of the 'puzzle' and his rejection, with which we concur, of two mooted solutions to it. (One of these uses standard, Kripkean, notions, and the other rests on work done by Arthur Prior.) We set out, and reject, the philosophy of modality underlying Fine's new solution, and we defend an alternative response to the alleged puzzle. Our solution follows the work of David Wiggins in distinguishing between the sentential operator 'It is necessary that' and the predicate modifier 'necessarily'. We briefly provide this distinction with a possible-world semantics on which it is neither a necessary truth, in some sense, that Socrates exists nor true, in some sense, that Socrates necessarily exists.

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. . . if tense logic is haunted by the myth that whatever exists at any time exists at all times, ordinary modal logic is haunted by the myth that whatever exists exists necessarily. (Prior 1957, 48)

1 A 'puzzle of possible non-existence'

(Fine 2005, 328) sets out 'a familiar puzzle concerning possible non-existence ... by means of the following argument':

(1) It is necessary that Socrates is a man.[1]

(2) It is possible that Socrates does not exist.

(3) Therefore it is possible that Socrates is a man and does not exist.

Now (Fine 2005, 329) remarks that 'the argument appears to be sound . . . yet its conclusion is unacceptable'. According to (Fine 2005, 329) the argument has the following valid logical form: $\Box p$, $\Diamond q : \Diamond(p \ \& \ q)$.

2 Two mooted, but deficient, responses

(Fine 2005, 329) rejects 'two responses to the puzzle that are implicit in the literature'. The first response, 'which', according to (Fine 2005, 329), 'derives from the framework of Prior's system Q', employs a modal logic with truth-value gaps. (Fine 2005, 329) continues that by its lights 'any proposition

[1]The sortal involved in the predicate 'is a man' is, in one sense (namely that in which a boy is not a man), a phased sortal rather than a substance sortal. (On the distinction, see, e.g., Wiggins 2001, 30.) In his discussion, however, Fine is concerned with the first sense of the noun 'man' listed in the *Oxford English Dictionary*: 'A human being, irrespective of age or sex'. In order to engage with his discussion without tampering with his terminology, we follow him in this. All uses of 'man' are to be taken to involve a substance sortal, rather than a phased one. Also, unlike Fine, when we evaluate the truth values of statements about Socrates, we will suppose (in order to avoid unnecessary complexities and philosophical commitments) that the time of evaluation is one at which Socrates is alive.

concerning an object is taken to be neither true nor false in any world in which the object does not exist'.[2] This gives rise to two notions of necessity. Where A is any truth-bearer, it is weakly necessary that A if and only if there is no possible world at which A is false. It is strongly necessary that A if and only if A is true at every possible world.[3] The Priorian response puts this distinction to use as follows. The argument from (1) and (2) to (3) is valid so long as the modalities are read weakly throughout. When we convince ourselves that (1) and (2) are true and that (3) is nevertheless false, this is because we slide from a weak reading of the modalities in (1) and (2) to a strong reading in (3). Read that way, the argument is invalid.

The second response, which (Fine 2005, *passim*) calls 'standard', and which might equally well be called 'Kripkean', invokes a distinction between 'qualified' and 'unqualified' necessity. (See Kripke 2011, 3, on weak necessity.) A statement of the form 'Fa' is unqualifiedly necessary if and only if it is true in all possible worlds.[4] It is qualifiedly necessary that Fa if and only if it is unqualifiedly necessary that if a exists, then Fa. This is applied to the argument from (1) and (2) to (3) as follows. The necessity involved in (1) is qualified. The possibility involved in (2) is unqualified. (3) is then invalidly concluded, with the sense of possibility involved in (3) being the unqualified sense. For this conclusion to follow, the sense of necessity involved in (1) would need to be the unqualified sense, which it cannot be if it is not necessary (in the unqualified sense) that Socrates exists. It is held that it is not an unqualified necessity that Socrates exists.

We agree with (Fine 2005, 330–4) that both the Priorian and the standard responses to the 'puzzle' are inadequate, not only as responses to the 'puzzle' but as philosophies of modality. Crucially, we agree with Fine that both responses are unable to deny 'that someone who accepts the necessity that Socrates is a man is thereby committed to accepting the necessity that Socrates exists' (Fine 2005, 331). We take it that this is a problem for the Priorian and standard accounts because it means that they fail to deliver on

[2] In order to avoid commitment to propositions, this could be reformulated as follows: a statement is neither true nor false if it contains an empty name.

[3] Appeal to possible worlds is not essential here. These notions can be reformulated as follows. It is weakly necessary that A if and only if it is absolutely impossible that A is false. It is strongly necessary that A if and only if it is absolutely impossible that A is not true. If there are truth-value gaps, then weak necessity so formulated is entailed by, but does not entail, strong necessity so formulated. The definitions of modal notions here are not, of course, reductive. It is questionable whether the appeal to possible worlds really has an advantage, in this respect, over a position that views modality as primitive.

[4] Fine formulates this in terms of propositions, not statements.

their philosophical desideratum that it is in no sense true that ‘whatever exists exists necessarily’. (In the case of the standard account, see further McLeod 2008.)

Let us show why Fine’s critique of the Priorian and standard responses to the ‘puzzle’ is correct.

Let ‘E’ be an existence predicate, ‘$\Box_W$’ be an operator for Priorian weak necessity, ‘$\Box_S$’ be an operator for Priorian strong necessity, ‘$\Box_Q$’ be an operator for standard qualified necessity and ‘$\Box_U$’ be an operator for standard unqualified necessity.

Let us first demonstrate the problem for the Priorian approach. Suppose that $\Box_W Fa$ & $\neg\Box_S Fa$. It follows from the latter conjunct that there is a possible world at which it is not true that Fa. It follows from the first conjunct that there is no possible world at which it is false that Fa. From these two consequences of the conjunction, it follows that there is a possible world at which it is neither true nor false that Fa. At that world, it is neither true nor false that Ea. So, it is not the case that $\Box_S Ea$. On the other hand, it follows that $\Box_W Ea$. This is because there is no world at which it is false that Ea, since the truth or falsehood of a statement at a world depends upon the non-emptiness, in that world, of the singular terms it contains. On the Priorian approach, then, it is necessary that Ea in just the same sense in which it is necessary that Fa. This is a problem for the Priorian approach because, as is clear from the words of Prior used as this article’s epigraph, Prior intended to reject the idea that there is a sense in which concrete objects exist necessarily.

The standard approach faces the same difficulty. Suppose that $\Box_Q Fa$ & $\neg\Box_U Fa$. It follows from the latter conjunct that there is a possible world at which it is false that Fa. It follows from the first conjunct, by the definition of qualified necessity, that there is no possible world at which it is false that $\neg Ea \vee Fa$. From these two consequences of the conjunction, it follows that there is a possible world at which it is false that Ea. So, it is not the case that $\Box_U Ea$. On the other hand, it follows that $\Box_Q Ea$. This is because there is no world at which it is false that $\neg Ea \vee Ea$. On the standard approach, then, it is necessary that Ea in just the same sense in which it is necessary that Fa. At least for Kripke, the originator of the standard approach, this is a problematic result. In a piece invoking weak necessity, (Kripke 2011, 15, note 11), Kripke is evidently committed to rejecting the idea that there is any sense in which it may correctly be said that whatever exists exists necessarily.

3 Fine's response

3.1 Matters worldly and unworldly

(Fine 2005, 321) contends that there is a distinction, analogous to 'a distinction between tensed and tenseless sentences', between worldly and unworldly sentences. On the back of this distinction, Fine alleges that the sorts of truths normally (and properly) regarded as necessary branch into two mutually exclusive kinds: 'the necessary truths proper ... and the transcendent truths' (Fine 2005, 321). Moreover, there are necessary existents, some of which are 'necessary existents proper' and others of which are 'transcendental existents' (Fine 2005, 321).

The suggestion is not that there are transcendent truths and transcendental existents over and above the necessary truths and necessary existents alleged to exist by some necessity-friendly metaphysicians. Rather, it is that necessity-friendly metaphysics usually overlooks some subtleties that are useful (and perhaps even required) for solving modal/existential puzzles and that are independently plausible (Fine 2005, 322, 328).

Let us set out Fine's distinctions and then turn to his attempt to apply them to the supposed modal/existential puzzle.

A truth-apt sentence that is not what Fine calls a 'hybrid sentence' is tensed if and only if 'it can properly be said to be true or false at a time' (Fine 2005, 322). Otherwise, it is tenseless. Fine's examples of tenseless sentences (Fine 2005, 322) are 'Socrates is a man' and 'Socrates is self-identical'; his example of a tensed sentence is 'Socrates does not exist'. The predicates 'is a man' and 'is self-identical' are in turn classified as tenseless, 'while a predicate such as "exists" is tensed' (Fine 2005, 322).

Now, says Fine, given that we have the distinction between tensed and tenseless sentences, and so between tensed and tenseless truths, we can make 'a corresponding distinction between sempiternal and eternal truths, a sempiternal truth being a tensed sentence that is always true and an eternal truth being a tenseless sentence that is true *simpliciter*.' In 'a restricted sense of truth-at-a-time' (Fine 2005, 323), only sentences true at some times and false at others, plus sempiternal truths, are true-at-a-time. In 'an extended sense of truth-at-a-time' (Fine 2005, 323) tenseless truths can be taken to be true at all times. The tenseless truths then qualify as being true-at-a-time, since truth at all times entails, given that there are times, truth at some times. On the extended conception of truth-at-a-time, 'an eternal truth will be true *regardless* of the time, i.e. regardless of how things are at the time, while a sempiternal truth will be [true] *whatever* the time, i.e. however things are at the time' (Fine 2005, 323). By way of examples, (Fine 2005, 323–4) remarks that 'If the sun will always shine, then "the sun will shine" will be true at any given time because of how things are at that time ... while the ... truth of "Socrates is [self-]identical" will not depend ... upon some ongoing feature of the universe'.

Let us now move from the tensed/tenseless distinction to the distinction Fine wishes to draw, by analogy with it, between worldly and unworldly matters. A truth-apt sentence that is not one of Fine's 'hybrid sentences' is *worldly* if and only if it depends for its truth-value 'upon the worldly circumstances' (Fine 2005, 321). Otherwise, it is unworldly. Among the truths that Fine takes to be normally and rightly regarded as necessary by philosophers, the necessary truths are true *whatever* the worldly circumstances, whereas the transcendental truths are true *regardless* of the worldly circumstances (Fine 2005, 324). Fine provides 'Socrates is self-identical' as an example of a truth that is unworldly and transcendental.[5] Earlier, it was cited by him as an example of a tenseless truth. In contrast with transcendental truths, necessary truths like 'Socrates exists or does not exist' depend upon the worldly circumstances. The example is a disjunction, and a disjunction depends for its truth on the worldly circumstances if and only if each of the disjuncts depends for its truth on the worldly circumstance (Fine 2005, 324, 326); each of 'Socrates exists' and 'Socrates does not exist' is a purely worldly sentence. Fine is supposing that if a set S contains only worldly sentences, then any compound formed only from elements of S and truth-functional operators is also a worldly sentence. This entails that a worldly atomic sentence makes any truth-functional compound containing finitely many instances of that sentence and no other atomic sentences worldly too.

Fine's next step is to propound an 'extended' sense of truth-at-a-world to parallel his earlier 'extended' sense of truth-at-a-time. Just as there he had taken the tenseless truths to be true at all times, under the extended notion of truth-at-a-time, so now he takes the transcendental truths to be true at all possible worlds, under the extended notion of truth-at-a-world. In

[5]For Fine, the notions of transcendental truth and of purely unworldly truth are co-extensive.

the extended sense of 'necessary' the transcendental truths may be said to be necessary, even though in the restricted sense of 'necessary' Fine distinguishes between the transcendental truths and the necessary truths proper. There is, however, a further extension here whose application to temporality is not canvassed by Fine, and that is the 'superextended' sense of truth-at-a-world. This applies to 'all propositions whatever' (Fine 2005, 326); thus, it applies not just to the worldly sentences and the unworldly sentences, but also to hybrid sentences, that is sentences 'that are composed of both worldly and unworldly components' (Fine 2005, 326). An example would be 'Socrates does not exist or is self-identical'. This sentence is a hybrid sentence, since it is a disjunction of a worldly sentence and an unworldly sentence, and is necessary in the super-extended sense, but not in the extended sense or the restricted sense.[6]

3.2 Fine's application of the worldly/unworldly distinction to the 'puzzle'

Recall that the apparent puzzle arises out of the following argument:

(1) It is necessary that Socrates is a man.

(2) It is possible that Socrates does not exist.

(3) Therefore it is possible that Socrates is a man and does not exist.

The supposed puzzle consists, in the words of (Fine 2005, 329), in the fact that 'the argument appears to be sound . . . yet its conclusion is unacceptable'. For Fine, solving the 'puzzle' consists not only in pronouncing on whether the argument really is sound or not, but also in explaining why our intuitions are pulled in different ways. So, how are Fine's distinctions meant to help solve the supposed puzzle?

Fine suggests two hypotheses that are fundamental for his solution: (i) the hypothesis that 'we are naturally inclined to use the modalities in an

[6]For more details about Fine's conceptions of restricted, extended, and super-extended modalities, see the Appendix to this article.

unextended or extended sense, though not in a superextended sense' (Fine 2005, 335); (ii) the hypothesis that "'exists" is a worldly predicate while such predicates as "man" or "identity" are unworldly' (Fine 2005, 336). So, says (Fine 2005, 336),

> [w]hen it comes to evaluating the first premiss [...], we implicitly treat 'man' as an unworldly predicate (in conformity with the [second] hypothesis). We then accommodate the sense of 'necessity' to the sense of 'man' by taking it in an extended sense (by the first hypothesis) and are thereby led to accept the first premiss.

Moreover, Fine thinks we are correct so to do, since he thinks 'is a man' *is* an unworldly predicate, and so 'Socrates is a man', since it is true, is a transcendental truth, and therefore an extended necessity. In evaluating the second premise, 'we implicitly treat "exists" as a worldly predicate (in conformity with the [second] hypothesis). We then take "possibility" in the unextended sense (by the first hypothesis) and are thereby led to accept the second premiss' (Fine 2005, 337). Again, we are correct so to do, since 'exists' *is*, according to Fine, in this context, a worldly predicate, and so, given that, intuitively, it is possible that Socrates does not exist, the sense of 'possible' here must be the restricted or unextended sense.

Now, for the dénouement. By Fine's own earlier definitions, the right thing for us to do is to take 'Socrates is a man and does not exist' to be a hybrid sentence, since 'is a man' is a transcendental predicate and 'does not exist' is a worldly one. On this account the right way for us to understand the use of 'possible' in (3) would then be as a super-extended use, and (3) would then come out as true. Fine has to explain why we intuitively think that (3) is false. He says that this is because by his first hypothesis we are 'disinclined' to use the super-extended sense of 'possible'; when we evaluate the conclusion we now take 'is a man' to be worldly (and so 'possible' in the restricted or unextended sense), contrary to how we took it in the first premiss. Fine explains that 'under this option, we take the possibility that Socrates is a man and does not exist to consist in the existence of a possible worldly circumstance in which Socrates is a man and does not exist' (Fine 2005, 337), and it is this that we rightly reject. Why do we (wrongly, by Fine's lights) take it to consist in the existence of that *worldly* circumstance? Fine seems to think that we are distracted by the presence of the word 'exist' in (3) into confusing the transcendental predicate 'is a man' with the worldly predicate 'is an existent man'. (To those worried that 'is an existent man' ought to be a hybrid predicate rather than a worldly one, Fine replies 'we may use the worldly predicate ["existent"] to qualify the unworldly predicate ["is a man"] [...] and thereby form a *worldly* predicate [that] is true in each world of those objects that are in the world-free extension of the first predicate and in the world-bound extension of the second predicate' (Fine 2005, 337).) On our incorrect, worldly, understanding of 'is a man' the premisses do not entail the conclusion, since the argument is now of the invalid form: $\Box_T p$, $\Diamond_R q$: $\Diamond_R(p \,\&\, q)$.

On the correct, unworldly, understanding of 'is a man', however, since 'Socrates is a man and does not exist' is a hybrid sentence, the sense of 'possible' involved is the super-extended sense. Since Fine thinks that we are already committed to the interpretation that the conclusion uses the super-extended sense of 'possible', he suggests that we should 'use the modalities in a superextended sense throughout the argument', in which case 'the conclusion does indeed follow from the premisses and should therefore be accepted' (Fine 2005, 338). On this hypothesis the argument contains no false premiss and is of the valid form $\Box_{SE} p$, $\Diamond_{SE} q$: $\Diamond_{SE}(p \,\&\, q)$.

Having set out the alleged puzzle, two deficient responses to it and Fine's own response, in the following two sections we turn to the evaluation of Fine's response and present our own response.

4 Dissolving the 'puzzle'

We hold, contrary to (Fine 2005, 328), that it is not the case that the 'puzzle' concerning possibility and non-existence 'would otherwise be quite baffling' without our having Fine's worldly/unworldly distinction at our disposal to help us solve it. (Fine 2005, 328) holds that the application of the worldly/unworldly distinction to solve the 'puzzle' counts as 'an important point in its favour'. While we do not argue against it here, the worldly/unworldly distinction appears to us, like Fine's version of the tensed/tenseless distinction, to be contentious. In fact, although we do not develop this point in what follows, we doubt that Fine has succeeded in articulating any genuine distinction between matters worldly and unworldly. We will now argue in favour of what we think is a better approach to the supposed puzzle than those we have discussed so far.

On our account, the problem with the argument from (1)–(3) is a lot simpler than the switch between different understandings of necessity put forward by the responses discussed so far (Priorian, standard and Finean).[7]

We begin by distinguishing, after (Wiggins 2001, 111–4), between two sorts of sentence that feature in natural language: statements of necessary truth, like

(1) It is necessary that Socrates is a man.

and statements of *de re* necessity like

(1*) Socrates is necessarily a man.

The modal operator in (1) is an operator on sentences. That in (1*) is embedded within a complex predicative expression and modifies a simpler predicative expression. We assume, after Wiggins, that statements of *de re* necessity are not generally reducible to statements of necessary truth. We also assume, again after Wiggins, that a true statement of *de re* necessity, when it is about a contingent being and contains no operator other than the modal operator, differs in truth value from its *de dicto* counterpart (which is, in virtue of the contingency of the contingent being's existence, false).

According to this approach, modal logics that employ only sentential modal operators cannot therefore adequately capture the distinction between modality *de dicto* and modality *de re* (cf. Wiggins 1976, 294). In order to illustrate this distinction formally, Wiggins employs the device of lambda abstraction, which is designed to create a predicate into which predicate modifiers can easily be inserted, and the predicate-modifier 'NEC'.[8]

(1) It is necessary that Socrates is a man.

$\Box[(\lambda x)[\text{Man } x], \langle \text{Socrates} \rangle]$

[7]Since (Fine 2005, 338) ultimately deems the argument sound, he thinks that the problem mainly lies not in the argument itself but in our intuitive analysis of it. This marks a difference between his approach, on the one hand, and, on the other, the Priorian approach, the standard approach and our own.

[8]While it is convenient to Wiggins's discussions and to ours to describe *de re* 'necessarily', and 'NEC', as predicate-modifiers, (Wiggins 2001, 111) expresses neutrality over whether these expressions really modify predicates or rather the copula. For a discussion of these options, and a defence of the view that modal adverbs are copula modifiers, see (McGinn 2000, Chapter 4).

(1*) Socrates is necessarily a man.

[NEC(λx)[Man x]], ⟨Socrates⟩

Approaches to the *de re*/*de dicto* distinction that employ only sentential modal operators will have difficulty in justifiably securing the result that in statements of necessity *de re*, like (1*), the name does not occur in an opaque context (again, cf. Wiggins 1976, 294). It is therein, rather than in considerations about the scope of a sentential modal operator (which is thereby, at least as normally understood in the literature, an intensional, opacity-producing operator), that the distinction between *de re* and *de dicto* modal statements is genuinely to be identified.

Let us assume, with Fine, and as the surface grammar of its constituent sentences suggests, that the argument (1)–(3) employs only sentential modal operators.

We contend, contrary to (1), that it is not, in fact, necessary that Socrates is a man. We think this because, intuitively, it might have been the case that there never was such an object as the concretely existent philosopher Socrates, and, as (Rumfitt 2003, 466) puts it, 'an object's existence is a pre-condition for its being either true or false that the object is such-and-such'. (Rumfitt clearly means that it is a *necessary* pre-condition not a merely contingent one.) We know that Fine disputes Rumfitt's and our intuitions here, but we are not persuaded by his attempts (Fine 2005, 345) to explain how something can be a man without existing.

If, however, we are wrong about this, and (1) is correct, and it is indeed necessary that Socrates is a man for the reason that Fine gives, viz. that it is a transcendental truth that he is a man, then we affirm that in that case (2) is false. If it is transcendentally true that Socrates is a man then Socrates will enjoy, along with his worldly existence, an unworldly, transcendental form of existence, and, since this is transcendental, it will be impossible that he should not have enjoyed it. We know that Fine will reply that, although (2) can legitimately be taken as coming out false in this way, it is more natural to interpret it as being about worldly existence and therefore to take it to be true. All we can say is that we find implausible the proposed switch from transcendental necessity in (1) to worldly necessity in (2), and that strong reasons would be required to overcome this implausibility. Moreover, if we are right that the approach to modality that we propose disposes of the supposed puzzle, then the strong reasons to adopt Fine's switch of modalities between the first and the second premiss are lacking.

As we remarked in Section 3.2, Fine thinks that solving the 'puzzle' consists partly in evaluating the soundness of the argument and partly in explaining why our intuitions are pulled in different ways when we try to evaluate its soundness.[9] Fine presented the argument as one in which we have two intuitively true premisses which, by a valid rule of inference, entail a conclusion that strikes us as intuitively unacceptable.

We can explain the intuition that (1) is true by saying that, when (1) is (mistakenly) evaluated as true, it is being taken to be a notational variant of (1*). We take it that if sortal essentialism is a true metaphysical doctrine and 'man' is a substance sortal, then 'Socrates is necessarily a man' is indeed true. Sortal essentialism, does not, we have agreed with Wiggins, entail (1) itself.

Regarding (2), if both the modality and the negation are taken to be sentential, then the original supposed intuition that (2) is true can be left intact. After all, we have argued that the embedded statement in (1) (i.e. the statement 'Socrates is a man') is in no genuine sense alethically necessary and it follows from this, given its truth-aptness, that there is some sense in which its negation is alethically possible. Reluctance to accept (3) is, we have argued, entirely well-founded. Even if it is held, for the reasons we have just mentioned, that (1) and (2) are intuitively true, the attendant reluctance, nonetheless, to accept (3) does not require the employment of Fine's distinction between matters worldly and unworldly (and the distinctions he launches on its back).

We can now set up a dilemma concerning the interpretation of (1):

(1) It is necessary that Socrates is a man.

There are two possible interpretations of (1). The first is that, in line with its surface syntax, it is a statement that employs a sentential modal operator:

$$\Box[(\lambda x)[\text{Man } x], \langle \text{Socrates} \rangle]$$

The second is that it is to be interpreted as a notational variant of (1*) ('Socrates is necessarily a man'), which we formalized as:

$$[\textsc{nec}(\lambda x)[\text{Man } x]], \langle \text{Socrates} \rangle$$

[9] In discussion, Fine has emphasized to us the importance, in his estimation of the philosopher's remit, of the second part of this task.

Assume, for the first horn of the dilemma, the first interpretation, i.e., that (1) employs a sentential modal operator. In that case, as we suggested above (after Rumfitt), (1) entails

(4) It is necessary that Socrates exists.

$\Box[(\lambda x)[\text{Exists } x], <\text{Socrates}>]$

Lest this formalization should give the impression that we admit a primitive first-level existence predicate, we now provide what we take to be a notational variant of it:

$\Box[(\lambda x)[(\exists y) y = x], <\text{Socrates}>]$

This variant employs the well-known classical-logical device of using the existential quantifier and the identity sign to formalize singular existentials.[10] Throughout, when we use ‘Exists’ in lambda translations, we intend to employ it merely as an abbreviation, not as a primitive first-level predicate.

Let us return to the main business. Provided that the modalities involved in (1) and (2) are duals and that (2) also employs a sentential modal operator, and given that (1) entails (4), (1) and (2) form an inconsistent pair. Thus, no puzzle is generated: the argument is unsound since one of the premises is false.

Assume, for the second horn of the dilemma, that (1) is to be interpreted as a notational variant of (1*). As a result, the argument is no longer of the valid form $\Box p$, $\Diamond q : \Diamond(p \,\&\, q)$. Instead, the argument is now:

(1*) Socrates is necessarily a man.

$[\textsc{nec}(\lambda x)[\text{Man } x]], \langle\text{Socrates}\rangle$

(2) It is possible that Socrates does not exist.

$\Diamond\neg[(\lambda x)[\text{Exists } x], \langle\text{Socrates}\rangle]$

[10] A translation that is more complicated still, though reducible to ours, can be obtained by emulating the approach suggested by (Wiggins 2003, 486). Thus: ‘$\Box[[(\exists y)((\lambda x = \text{Socrates}], \langle y\rangle]$’ . This illustrates the ability of the lambda notation to reflect the grammatical distinction between ‘Socrates is identical with something’ (our formulation) and ‘Something is identical with Socrates’ (Wiggins 2003).

(3) Therefore it is possible that Socrates is a man and does not exist.

$\Diamond[(\lambda x)[\text{Man } x], \langle\text{Socrates}\rangle \ \& \ \neg[(\lambda x)[\text{Exists } x], \langle\text{Socrates}\rangle]]$

Thus, the argument is invalid, since the two premises employ different modalities: (1*) uses *de re* necessity, and does not entail its *de dicto* counterpart, while (2) uses *de dicto* possibility. Once again, there is no puzzle.

5 *De re* necessity and contingent existence

Our approach to dissolving the 'puzzle' involved appeal to a version of the distinction between necessity *de re* and *de dicto* that remains unorthodox. Relatedly, its technical details remain relatively under-developed. In particular, it might be objected that our approach does not seem to be readily compatible with model-theoretical approaches to the formal semantics of modal logics. Our response to this objection consists of two points. First, we note that the form of distinguishing between the logical syntax of *de re* and *de dicto* modal sentences that we have favoured, including the deployment of the lambda notation, has already been adopted by some authors who also, in fact, provide both types of sentence with model-theoretical semantics: (Fitting and Mendelsohn 1998, 85–8, 193, 201, 213, 217) and (Garson 2013, Chapter 19). Secondly, this section and the next one explain the distinction in more detail and summarize a model-theoretical semantics that preserves the judgements made in the foregoing sections about the truth values of relevant modal sentences.

We take it that if Socrates is a contingent being then it is neither necessary that Socrates exists nor the case that Socrates necessarily exists. Thus, we take both of the following sentences to be false:

(4) It is necessary that Socrates exists.

$\Box[(\lambda x)[\text{Exists } x], \langle\text{Socrates}\rangle]$

(4*) Socrates necessarily exists.

On the assumption that there is no primitive first-level existence predicate, and that 'Socrates exists' contains neither the copula nor such a predicate, the ‘necessarily’ in (4*) cannot be either a copula modifier or the modifier of such a predicate. On our view, given that Socrates is a contingent existent, (4*) admits of no true essentialist reading, and is significantly different from standard essentialist statements like (1*).

We think that someone unacquainted with formal systems of modal logic that took (1*) to be true would be unlikely to regard (4*) as true. We think that such a person would understand (4*) as asserting Socrates to be a necessary being, rather than merely necessarily being a being (or necessarily being an object). We do not intend this to be a merely pragmatic point. Rather, our point is that taking (4*) to admit of an interpretation on which it is true appears to us to be subversion of what the sentence means.[11]

Someone might offer this formalization of (4*), treating the sentence in broadly the same manner as (1*) above:

$$[\textsc{nec}(\lambda x)[(\exists y) y = x]], \langle \text{Socrates} \rangle$$

It seems to us that this formalization is apt for (the true) ‘Socrates is necessarily a being’, but is not apt for (the false) ‘Socrates is a necessary being’ and, thus, not apt for (the false) (4*) (i.e., ‘Socrates necessarily exists’).

Our position with respect to (4) and (4*) just spells out what we think is involved in holding that Socrates is a contingent, rather than (unlike such supposed cases as God, numbers, and propositions) a necessary, being. With respect to (4*), we agree with (Fine 2005, 329):

> There may be items—numbers or propositions or the like—that necessarily exist. But Socrates (and Felix) are not among them. Indeed, it is surely possible that no men (or cats) exist; and from this it follows, given the appropriate version of the first premiss, that it is possible that Socrates (or Felix) does not exist.

In denying that (1*) entails (4*), we part company with a philosopher whose general approach to the distinction between modality *de dicto* and *de re* we share, namely Efird.

[11] Also, when people that think that there are contingent beings suggest that (4*) is true, we believe that they interpret (4*) in a way that dilutes its real–and, in philosophy and theology, traditional–meaning. When theologians say that God necessarily exists they are not appealing to a notion of necessary existence that is *true of all objects*. Indeed, we deny that there is any such *bona fide* notion. The suggestion that there is some sense of ‘necessarily’ on which (4*) is true seems to us to stem either from the global denial of contingent existence or from a reconstruction of natural-language meaning in the light of the features of a formal system of modal logic.

Efird's approach combines a Wiggins-style approach to the *de re*/*de dicto* distinction with a Priorian semantics. It has the result (Efird 2010, 107), as we also contend, that there are no necessary truths about contingent existents. Thus, to use Efird's example, it is not the case that it is necessary that Timothy Williamson exists. Nevertheless, Efird's approach has the further result (Efird 2010, 97, 107), which ours does not, that Williamson necessarily exists.[12] According to (Efird 2010, 107), when the Priorian semantics is adopted:[13]

> we can [. . .] define a scheme for translating formulas containing 'nec' to sentences containing '$\Diamond$' and '$\neg$': [nec$(\lambda x)(P(x))$], $[a]=_{df.}$ $\neg\Diamond\neg P(a)$. This then yields the following [. . .]:
>
> > 'Necessarily, *a* is F' is represented formally as: $\Box$Fa.
> >
> > '*a* is necessarily F' is represented formally as: (Fa & $\neg\Diamond\neg$Fa).
> >
> > 'Contingently, *a* is F' is represented formally as: (Fa & $\neg\Box$Fa).
> >
> > '*a* is contingently F' is represented formally as: (Fa & $\Diamond\neg$Fa).
>
> These formalizations give rise to the consistent conjunctive schema:
>
> > '*a* is necessarily F and, contingently, *a* is F' is represented formally as: [Fa & ($\neg\Diamond\neg$Fa & $\neg\Box$Fa)].
>
> As an instance of this conjunctive schema we have: Williamson necessarily exists, and, contingently, Williamson exists [$\exists!a$ & ($\neg\Diamond\neg\exists!a$ & $\neg\Box\exists!a$)]. So [. . .] Williamson is, in a *de re* sense, a necessary existent, but he is not a necessary existent in the *de dicto* sense:

[12]We note that (Efird 2010, 106) assumes that Wiggins's account of *de re* 'necessarily' properly applies to modalized existentials like (4*), and appears to see no difficulties with this assumption. We have argued, in this section, that it does not so apply, and, in so doing, we have made such difficulties plain. Also, (Wiggins 1995, 2003) seems to favour treating 'exists', as we do, as a defined predicate. So, the proponent of a Wiggins-style approach to the *de re*/*de dicto* distinction does not seem to be forced to go down Efird's road of taking it that there is a sense in which 'Socrates necessarily exists' is true.

[13]Efird uses 'nec' in place of Wiggins's 'NEC', '$\exists!$' as an existence predicate, and square brackets both where Wiggins would use square brackets and in place of Wiggins's angled brackets.

Williamson necessarily exists, but it is not necessary that he exists.[14]

Importantly, for our purposes, Efird's approach has the consequence that it is true that Williamson necessarily exists. Efird has to affirm, then, that (1*) entails (4*); indeed, he embraces this consequence. On the interpretation of (4*) that Efird gives, (4*), though true, does not entail (4), and does not contradict (2). On Efird's account, just as each object is necessarily identical with itself, so each object necessarily exists. Let us set out, and explain the reasoning behind, our own position. Consider the following sentences:

(5) Socrates is necessarily self-identical.

$[\textsc{nec}(\lambda x[x = x]], \langle\text{Socrates}\rangle$

(6) Socrates is necessarily an existent.

$[\textsc{nec}(\lambda x)[\text{Existent } x]], \langle\text{Socrates}\rangle$[15]

(7) Socrates is necessarily an object.

$[\textsc{nec}(\lambda x)[\text{Object } x]], \langle\text{Socrates}\rangle$

(8) Socrates is a necessary existent.

$\Box[(\lambda x)[\text{Existent } x], <\text{Socrates}>]$

Supposing that (1*) is true, (5) follows, for whatever is necessarily a man is also necessarily self-identical. Likewise, and with similar reasoning, (6) and (7) also follow from (1*). While it is the case that whatever is necessarily a

[14] It is evident, then, that Efird holds that 'Williamson is a necessary existent' is ambiguous. In the main text, we deny that it is ambiguous. Moreover, even if, as Efird argues, 'Williamson necessarily exists, and, contingently, Williamson exists' is true, it does not follow immediately that there are two senses of 'Williamson is a necessary existent'. This is because that sentence employs neither a modal operator nor a modal adverb, but, rather, a modal adjective. We note, further, that non-existential sentences that employ 'necessarily' as a predicate modifier, like 'Socrates is necessarily human', do not seem equivalent to the corresponding sentences featuring the adjective 'necessary', like 'Socrates is a necessary human': it is questionable whether these transformations yield sentences that are well-formed and semantically complete, let alone ones that mean what the source sentences mean. This seems to be a difference between modalized non-existentials and modalized existentials, like 'Socrates necessarily exists', which *do* seem equivalent to the corresponding sentences featuring the adjective 'necessary' (in this case, 'Socrates is a necessary existent').

[15] In this formalization, 'Existent' is not to be read adjectivally. Compare 'Object' in (7).

man is necessarily an object, and necessarily an existent, it is not the case that whatever is necessarily a man is a necessary object or a necessary existent. Thus (8), unlike (5)–(7), does not follow from (1*). While (8) entails that it is necessary that there is such an existent as Socrates, (6) and (7) do not.

(4*) might at first appear to admit of two different readings, namely (6) and (8). Nevertheless, we see no good reason to view (6) as an admissible reading of (4*). While whatever exists is necessarily a being, an object, or an existent, this does not entail that whatever exists exists necessarily: thus, (6) is logically weaker than (4*). If, as we believe, (6) is an incorrect reading of (4*), then (8) is the correct reading.[16] The distinction between (6) and (8) seems as clear to us as that between the true 'Socrates is necessarily a being' and the false 'Socrates is a necessary being'. Since (8) admits of no ready reading on which it is true, and since (6) is not available as an independently plausible, rather than theoretically forced, reading of (4*), (4*) also admits of no ready reading on which it is true.

6 Semantics for *de re* necessity with contingent existence

While it might be thought that a Wiggins-style approach to the syntax of *de re* modality is not readily reconcilable with the standard model-theoretical (or 'possible worlds') semantics for modal logics, we noted, at the beginning of the previous section, that this is not really the case. What *has* been lacking in the literature so far, however, is a semantics that secures the intuitive judgements about the truth values of modal sentences that partly motivate the Wiggins-style approach, and that is a semantics consistent with the judgement that there is no sense of 'necessarily' in which 'whatever exists exists necessarily' is true.

We now specify a semantics fit to accommodate the attribution of *de re*

[16]We take this to be so independently of whether the lambda translation given for (8) is ultimately correct. One reason for doubt here is that in sentences like 'Socrates necessarily exists', the alternatives we have explored leave us with no method of formalization that both captures the logical strength of the sentence and allows 'necessarily' to remain, as it appears, a referentially transparent operator. Also, (8) features not a modal operator or adverb, upon which modal logics have overwhelmingly concentrated, but a modal adjective. The logical behaviour of modal adjectives is neither studied as often nor understood as well. There seems to be work left to do in resolving these difficulties.

necessary attributes to beings that are held to exist contingently.[17] Our semantics has the following features.

First, no atomic sentence is a logical truth. In classical quantificational logics, any identity sentence of the form $\ulcorner c = c \urcorner$ is a logical truth. In classically-based quantified modal logics, any formula of the form $\ulcorner \Box c = c \urcorner$ is normally a logical truth. (For an exception, see Nelson 2016.) Since, on our account, no atomic sentence is a logical truth, we avoid the result that, for any arbitrarily selected object, it is a true statement of *de dicto* necessity that the object is identical with itself.

Secondly, our semantics is that of a universally free negative free logic. That is to say, as well as employing free-logical quantification (Garson 2013, Ch. 13, esp. 264), and permitting both empty names and an empty domain, we treat every atomic sentence containing an empty name as thereby false.

Thirdly, the presence of empty names, does not, on our semantics, give rise to trivially true statements of *de re* necessity. For example, it is a logical truth, because it is an instance of the law of identity of propositional logic, that if Pegasus exists then Pegasus exists. As a consequence, when the weak necessity account of *de re* necessity is adopted, and extended to natural language, Pegasus necessarily exists, and, indeed, necessarily—albeit trivially so—has every property whatsoever, as a matter of weak necessity—see further (McLeod 2008, 319).

Fourthly, essentialist facts about contingent beings, if there be such facts, are metaphysical: they are not logical truths. They have the same logical status as atomic facts. Thus, the *de re* necessitation of an atomic sentence, which involves the insertion of a predicate-modifying 'necessarily' into the sentence (as when 'Socrates is human' is turned into 'Socrates is necessarily human') will have a result that, like the simpler sentence, is neither logically true nor logically false.

We now spell out the semantic differences between, (1) and (1*); we then provide general semantic clauses for the modal expressions in these sentences. In what follows, following (Garson 2013), '$\mathbf{wRv}$' means that the possible world $\mathbf{w}$ has access to the possible world $\mathbf{v}$ (in other words, $\mathbf{v}$ is accessible from $\mathbf{w}$). '$\mathbf{a_w}$(It is necessary that Socrates is a man) = T', for example, means that the sentence in parentheses is true at the world $\mathbf{w}$.[18]

[17] We do not here attempt fully to specify a formal system fit for this task. We hope to do so elsewhere.

[18] Our semantic clause for 'It is necessary that' mirrors that of '$\Box$' in the modal logic **K**: see (Garson 2013, 64). That for predicate-modifying 'necessarily' is different.

Semantics for (1):

> $\mathbf{a_w}$(It is necessary that Socrates is a man) = T if and only if, for each possible world, $\mathbf{v}$, such that $\mathbf{wRv}$, $\mathbf{a_v}$(Socrates is a man) = T.

Semantics for (1*):

> $\mathbf{a_w}$(Socrates is necessarily a man) = T iff $\mathbf{a_w}$(Socrates exists) = T, and for each possible world, $\mathbf{v}$, such that $\mathbf{a_v}$(Socrates exists) = T and $\mathbf{wRv}$, $\mathbf{a_v}$(Socrates is a man) = T.

Semantics for 'It is necessary that':

> $\mathbf{a_w}$(It is necessary that c is A) = T iff for each $\mathbf{v}$ such that $\mathbf{wRv}$, $\mathbf{a_v}$(c is A) = T.

Now let A be a primitive predicate (that is, one that is not defined using any item, other than the identity sign, from the logical vocabulary).[19] With this restriction in place, we now specify our semantics for predicate-modifying 'necessarily'.

Semantics for predicate-modifying 'necessarily':

> $\mathbf{a_w}$(c is necessarily A) = T iff $\mathbf{a_w}$(c exists) = T and for each $\mathbf{v}$ such that $\mathbf{a_v}$(c exists) = T and $\mathbf{wRv}$, $\mathbf{a_v}$(c is A) = T.

On these semantics, (1), but not (1*), entails that it is necessary that Socrates exists. Moreover, (1*) does not entail, in any *bona fide* sense of 'necessarily', the sentence 'Socrates necessarily exists'. Our semantics for predicate-modifying 'necessarily', though simple, is intended to achieve the breakthrough of providing for a Wiggins-style account of the *de re*/*de dicto* distinction a

[19]While this restriction may go further than is strictly necessary in that it excludes *all* defined predicates, it is motivated by the case of 'exists', as discussed in Section 5 above. The discussion there was intended, in part, to show that the exclusion of 'exists', at least, from the clause for predicate-modifying 'necessarily' was not an arbitrary restriction.

model-theoretical semantics that allows there to be true statements of *de re* necessity about contingent beings, without having the result that, in some sense of 'necessarily', the statement 'whatever exists exists necessarily' is true.

The defined status of the existence predicate means that 'Socrates necessarily exists', for which our semantic clause for predicate-modifying 'necessarily' does not account, must be parsed in one of the following ways:

(9) It is necessary that there is something with which Socrates is identical.

(10) There is something that is necessarily identical with Socrates.

On the first parsing, which is the one that both natural language and traditional philosophical and theological parlance about necessary beings would suggest to be appropriate, there is (unlike in cases that feature a primitive monadic predicate) no semantic difference between 'Socrates necessarily exists' and 'It is necessary that Socrates exists'. (Given that it is not the case that Socrates exists in all possible worlds, both sentences are false.) Parsed in the second way, 'Socrates necessarily exists' is true, but its surface form very much belies both its logical form and its semantics. Moreover, the proposal that the sentence ought to be so parsed is one for which we see no good rationale and against which we have already argued in Section 5.

7 Conclusion

We conclude that the argument from (1)–(3) is unsound, and that it does not generate any deep puzzle. The predominant reason given by Fine for subscribing to the worldly/unworldly distinction is as a means of solving the 'puzzle', with any other reasons being only alluded to, rather than spelled out, in his piece. Thus the distinction is, on the evidence Fine presents, undermotivated. In the case of the supposed puzzle, at least, it is unnecessary. Also, and perhaps more significantly, a Wiggins-style approach to the distinction between *de re* and *de dicto* modality not only dissolves Fine's puzzle, but can be provided with a formal semantics that secures some of the philosophical motivations of that approach. These include reconciling true statements of *de re* necessity about concrete objects with the denial that there is a sense of 'necessarily' in which the statement 'whatever exists exists necessarily' is true.

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8 Appendix: Fine on modalities

8.1 Grades of modality

The purpose of this technical appendix is to further our understanding, and that of the reader, of the distinctions between modal notions that Fine makes in his article and to develop some details left aside in it. (This is despite our own scepticism about the need for, and indeed desirability of, these distinctions.) While we acknowledge that it would be useful to do this for quantified modal logic, we restrict our attention to sentential modal logic.

Recall that Fine distinguishes between restricted and extended modalities. Let '$_R$' be a subscript for restricted modality, '$_T$' be a subscript for transcendental modality, and '$_E$' be a subscript for extended modality. Where A is any truth-valued sentence *that is either purely worldly or purely unworldly*,[20] we shall now define restricted, transcendental, and extended notions of necessity and possibility. We take our definitions to spell out those of (Fine 2005, 325–7), although ours differ from his in that ours do not place the same emphasis

[20]Such sentences contrast with the 'hybrid' sentences with which we are concerned in Section 8.2 below.

on possible worlds. These definitions are not of operators in a formal language, as box and lozenge are normally taken to be, but instead are definitions of meta-linguistic predicates attached to names of sentences, after (Quine 1953): 'Nec' and 'Poss' are short for 'is necessarily true' and 'is possibly true'.

(Def Poss_R) $\mathrm{Poss}_R A =_{df}$ there is some worldly circumstance, c, such that $\mathbf{a_c}(A) = \mathrm{T}$.[21]

(Def Nec_R) $\mathrm{Nec}_R A =_{df} \mathrm{Poss}_R A$ & $\neg\mathrm{Poss}_R\neg A$ (i.e., for every worldly circumstance, c, $\mathbf{a_c}(A) = \mathrm{T}$).

(Def Nec_T) $\mathrm{Nec}_T A =_{df}$ A is true & $\neg\mathrm{Poss}_R A$ (i.e. A is true and yet there is no worldly circumstance, c, such that $\mathbf{a_c}(A) = \mathrm{T}$).

(Def Poss_T) $\mathrm{Poss}_T A =_{df} \mathrm{Nec}_T A$.[22]

(Def Nec_E) $\mathrm{Nec}_E A =_{df} \mathrm{Nec}_R A \vee \mathrm{Nec}_T A$.

(Def Poss_E) $\mathrm{Poss}_E A =_{df} \mathrm{Poss}_R A \vee \mathrm{Poss}_T A$.

8.2 Super-extended modalities

(Fine 2005, 326) asserts that the idea of super-extended truth (SE-truth) (and so that of super-extended necessity (SE-necessity)) 'applies to "hybrid" propositions. These are propositions . . . that are composed of both worldly and unworldly components. Their truth-value in a given world turns partly on the worldly facts and also partly on the transcendental facts'.[23]

Fine also allows for (i) hybrid predicates, i.e. predicates containing at least one worldly part and at least one unworldly part, e.g. 'is existent and is a man'; cf. (Fine 2005, 338), and (ii) hybrid terms, i.e. terms containing at least

[21] The restriction to worldly circumstances is necessary so as to rule out transcendental truths.

[22] The notions of transcendental truth, transcendental necessity and transcendental possibility are co-extensive.

[23] As may already be evident, we prefer to work with the idea of the sentence rather than that of the proposition.

one worldly part that is semantically significant and at least one unworldly part that is semantically significant, e.g. 'the singleton set of Socrates'.[24]

Despite his allowing for these two categories, we shall presume that, for Fine, any hybrid sentence must be truth-functionally complex, i.e. that there are no atomic hybrid sentences. This is for four reasons. First, it is simpler to analyse logically complex predicates in a truth-functional way, i.e. it is simpler to analyse 'Socrates is existent and is a man' (for example) not as an atomic sentence featuring the single predication of a conjunctive predicate, but rather as a conjunction of two sentences each featuring a single predicate predicated of an identical subject, thus 'Socrates is existent and Socrates is a man'. Secondly, Fine holds that compound predicates in which one element qualifies the other are not hybrid, thus 'existent man' (for example) is a worldly predicate, not a hybrid one (Fine 2005, 337). Thirdly, Fine holds that sentences containing a hybrid name are not on that account hybrid sentences: their status depends on the status of the predicates that they contain; thus, 'the singleton of Socrates is self-identical', for example, is transcendentally true. Fourthly, for Fine, a sentence with a worldly term and an unworldly predicate, e.g. (assuming that 'Socrates' is a worldly term) 'Socrates is self-identical', is unworldly, and a sentence with an unworldly term and a worldly predicate, e.g. 'the number three is being thought about by Socrates', is worldly.

Now let A be any compound sentence in a truth-functional formal language, or in a truth-functional fragment of a natural language. We now define what it is to be a *hybrid sentence* of such a language.

Case i. Where ζ is a dyadic operator and A is a sentence of the form $B \zeta C$, A is a hybrid sentence if and only if one of the following obtains:

(i) either B or C is a hybrid sentence;

(ii) B is a worldly sentence and C an unworldly sentence, or vice versa.

Case ii. Where A is of the form $\neg B$, A is a hybrid sentence if and only if B is a hybrid sentence.

This concludes our definition of *hybrid sentence*. Let us now set out a principle that we shall then use to set up an inductive definition of the super-extended

[24] (Fine 2005) does not actually discuss hybrid *terms*, but does discuss hybrid *objects* using the singleton of Socrates as an example (Fine 2005, 352).

modalities (SE-modalities).

> **(Main Principle)(MP)** If A is extendedly necessary (E-necessary) then A is super-extendedly necessary (SE-necessary).

If A is atomic then (MP) is the only way in which A can be SE-necessary, since no hybrid sentence is atomic. Since A is E-necessary if and only if $\neg A$ is extendedly impossible (E-impossible), and A is SE-necessary if and only if $\neg A$ is super-extendedly impossible (SE-impossible), (MP) entails:

> **(Supplementary Principle)(SP)** If A is E-impossible then A is SE-impossible.

If A is atomic then (SP) is the only way in which A can be SE-impossible, since no hybrid sentence is atomic.

> **(SE-equivalence)(SE)** If two contingent (worldly) sentences have the same truth-value as each other across possible worlds then they are SE-equivalent; if two worldly or transcendent sentences are both E-necessary then they are SE-equivalent, and if two worldly or transcendent sentences are both E-impossible then they are SE-equivalent. No other two sentences are SE-equivalent.

We also need to make use of the following theorem, which deploys the notion of SE-equivalence:

> **Substitution Theorem** Let $A(B)$ be a sentence containing an occurrence of B. $A(B)$ is then SE-equivalent to $(B \,\&\, A(C)) \vee (\neg B \,\&\, A(C))$ where C is SE-equivalent to B.

In other words, take $A(B)$. Find a sentence, C, that is SE-equivalent to B and substitute it for B. Conjoin this new sentence with B. This gives $B \,\&\, A(C)$ (call this 'sentence 1'). Then negate B to get $\neg B$. Conjoin the result with $A(C)$. This gives $\neg B \,\&\, A(C)$ (call this 'sentence 2'). Now disjoin sentence 1 with sentence 2 to get $(B \,\&\, A(C)) \vee (\neg B \,\&\, A(C))$. This can easily be seen to be equivalent to $A(B)$. (Proof: suppose (i) that both $A(B)$ and B are true, then the first disjunct, sentence 1, will be true. Suppose (ii) that $A(B)$ is true and B is false, then the second disjunct, sentence 2, will be true. Suppose (iii) that $A(B)$ is false and B is true, then both disjuncts, sentence 1 and sentence 2, will be false. Suppose (iv) that both $A(B)$ and B are false, then again both disjuncts, sentence 1 and sentence 2, will be false.)[25]

We know that every hybrid sentence contains as a proper part at least one unworldly sentence. These unworldly sentences will play the role of B above. We make continued use of the substitution theorem until every embedded unworldly sentence in A has been removed and replaced with an SE-equivalent *worldly* sentence (e.g. 'Socrates exists or does not exist' if the unworldly sentence is transcendentally true, and 'Socrates exists and does not exist' if the unworldly sentence is transcendentally false)—these sentences will play the role of C above.

This means that we are left with a (possibly embedded) sentence of the form $(B \mathbin{\&} A(C)) \vee (\neg B \mathbin{\&} A(C))$ in which B is unworldly and C is a worldly equivalent of B. It remains for us to define when this sentence, $(B \mathbin{\&} A(C)) \vee (\neg B \mathbin{\&} A(C))$, is SE-necessary.

Suppose (i) that B is transcendentally true. It follows that C is a worldly necessity. It further follows, by (MP) and (SP) above, that B and C are SE-necessary. We now need to evaluate each of the disjuncts of $(B \mathbin{\&} A(C)) \vee (\neg B \mathbin{\&} A(C))$. Let us start with the first disjunct. Since $A(C)$ is not a hybrid sentence—all of its component sentences are purely worldly—we already know, by (MP) and (SP) above, the modal status of $A(C)$: if $A(C)$ is E-necessary then it is SE-necessary, if it is E-impossible then it is SE-impossible, and if it is E-contingent then it is SE-contingent. Since B is SE-necessary, and a conjunction is SE-necessary if and only if each conjunct is SE-necessary (since, by its truth-table, a conjunction is true when and only when both its conjuncts are true), it follows that if $A(C)$ is E-necessary then $B \mathbin{\&} A(C)$ is SE-necessary, if $A(C)$ is E-impossible then $B \mathbin{\&} A(C)$ is SE-impossible, and if $A(C)$ is E-contingent then $B \mathbin{\&} A(C)$ is SE-contingent. But since the second disjunct is SE-impossible (since $\neg B$ is SE-impossible), irrespective of $A(C)$, it follows, further, that if $A(C)$ is E-necessary then $B \mathbin{\&} A(C) \vee (\neg B \mathbin{\&} A(C))$ is SE-necessary, if $A(C)$ is E-impossible then $(B \mathbin{\&} A(C)) \vee (\neg B \mathbin{\&} A(C))$ is SE-impossible, and if $A(C)$ is E-contingent then $(B \mathbin{\&} A(C)) \vee (\neg B \mathbin{\&} A(C))$ is SE-contingent.

Suppose (ii) that B is transcendentally false. It follows that C is a worldly impossibility. It further follows, by (MP) and (SP) above, that B and C are

[25] We are very grateful to Kit Fine for drawing our attention to this.

SE-impossible. We need, again, to evaluate each of the disjuncts of $(B \,\&\, A(C)) \vee (\neg B \,\&\, A(C))$. Let us start with the second disjunct. Again, by (MP) and (SP) above, we already know the modal status of $A(C)$, since it is not a hybrid: if $A(C)$ is E-necessary then it is SE-necessary, if it is E-impossible then it is SE-impossible, and if it is E-contingent then it is SE-contingent. But, further, since $\neg B$ is transcendentally true (since B was transcendentally false), it follows that if $A(C)$ is E-necessary then $\neg B \,\&\, A(C)$ is SE-necessary, if $A(C)$ is E-impossible then $\neg B \,\&\, A(C)$ is SE-impossible, and if $A(C)$ is E-contingent then $\neg B \,\&\, A(C)$ is SE-contingent. But since the first disjunct is SE-impossible, since B is SE-impossible (and a conjunction is SE-impossible if and only if at least one conjunct is SE-impossible (since, by its truth-table, a conjunction is false when and only when at least one conjunct is false)), it follows that if $A(C)$ is E-necessary then $(B \,\&\, A(C)) \vee (\neg B \,\&\, A(C))$ is SE-necessary, if $A(C)$ is E-impossible then $(B \,\&\, A(C)) \vee (\neg B \,\&\, A(C))$ is SE-impossible, and if $A(C)$ is E-contingent then $(B \,\&\, A(C)) \vee (\neg B \,\&\, A(C))$ is SE-contingent.

It may be that $(B \,\&\, A(C)) \vee (\neg B \,\&\, A(C))$ is itself an embedded sentence, in which case we just repeat the procedure now that we know the modal status of $(B \,\&\, A(C)) \vee (\neg B \,\&\, A(C))$. Since every compound sentence will be covered by repeated applications of this procedure, and since no atomic sentence is a hybrid sentence, this concludes our recursive definition of SE-modality.

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RESEARCH ARTICLE

Wiredu contra Lewis on the Right Modal Logic

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Abstract: This paper is a critical study of an argument put forward by Kwasi Wiredu in his engagement with C. I. Lewis on choosing the right modal logic for logical necessity. Wiredu argues that Lewis "could have been more adventurous modally with perfect logicality" and could justifiably have accepted S4 rather than being "to the last cautious of any system stronger than S2" (Wiredu 1979). I address terse, incomplete, and provocatively incongruous notes on Wiredu's paper by (Makinson 1980) and (Humberstone 2011), as well as a paper by (Cresswell 1965) that Humberstone cites, and I draw on recent work by (Lewitzka 2015; 2016). I conclude that Wiredu's argument cannot be accepted as sound but a variant argument can be accepted as sound.

Keywords: C. I. Lewis; equivalence; identity; Kwasi Wiredu; modal logic; S4.

1 Introduction

Kwasi Wiredu argues against C. I. Lewis that the right modal logic for logical necessity is not S2 but "(at least) S4" (Wiredu 1979, 692; see Lewis

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and Langford 1959). My questions are, what is Wiredu's argument exactly, and is it sound? I conclude that Wiredu's argument cannot be accepted as sound but a variant argument can be accepted as sound.

My questions are prompted by the provocative incongruity between, and incompleteness of, terse notes on Wiredu's paper by David Makinson and Lloyd Humberstone. Incongruously, while Makinson finds "a technical error" that he says "vitiates both parts of the paper" and "some confusion" in the second part (Makinson 1980), Humberstone credits Wiredu's paper with a worthwhile "observation" and "result" that Humberstone also credits to a paper by Max Cresswell (Humberstone 2011, 604, 1154; see Cresswell 1965, not cited by Wiredu). Moreover, Makinson's and Humberstone's notes are limited to only some formal aspects of Wiredu's paper, where the main concern really is pragmatic in a sense that borders on epistemic. (The broader philosophical contexts of Wiredu's and Lewis's formal investigations are sketched by Osha 2014 and Hunter 2016.) A satisfying treatment of Wiredu's paper thus should address the incongruity and the incompleteness of Makinson's and Humberstone's notes, and the relation of Wiredu's argument to Cresswell's.

As suggested by Makinson's comments quoted above, Wiredu frames his paper as providing two arguments for his main conclusion, one primary and the other "alternative" (Wiredu 1979, 692). The alternative argument is aimed at showing that "the addition of [a certain] definition of necessity [. . .] to a suitable system of propositional logic yields at least S4" (Wiredu 1979, 693). It seems to me that Humberstone is silent about Wiredu's primary argument and that his attention is focused on formal aspects of this alternative argument, which he takes to be in relevant respects essentially the same as Cresswell's argument, and which he thinks can be understood in such a way as to be sound and free from any insuperable technical error or confusion of the sort Makinson reports. I think in the end, though, Wiredu's paper is best understood as providing a single argument for his main conclusion, but with two formally distinct lines of support for a key premise in that argument.

My attention for the most part is focused on what Wiredu presents as his primary argument, where Wiredu's pivotal challenge to Lewis is this.

> Lewis himself was to the last cautious of any [modal] system stronger than S2 . . . . [But] he could have been more adventurous modally with perfect logicality . . . . [For there is] a way of establishing the logical inevitability of S4 [using a certain substitution rule that

> is] very plausible and intuitively acceptable .... [When either S1 or S2 is appropriately extended with that rule,] (at least) S4 becomes available .... [So,] by Lewis' own logical interpretation of the modalities he was committed to S4 ... (Wiredu 1979, 689, 691, 692, 694).

Clearly, while Wiredu thinks S4 is the right modal logic—in the sense that the truth about logical necessity includes at least S4—that is not exactly what he intends to argue. Rather, his challenge is a circumstantial *ad hominem* to Lewis. Wiredu's argument in outline is that, because Lewis justifiably accepted that the truth about logical necessity includes at least S1 or S2, and Lewis could justifiably have accepted the substitution rule Wiredu has in mind, and S4 is deductively contained in the system that results when either S1 or S2 is appropriately extended with that substitution rule, therefore Lewis could justifiably have accepted that the truth about logical necessity includes at least S4. Fairly interpreting and assessing Wiredu's argument thus requires considering exactly what rule he has in mind and exactly why he thinks Lewis could justifiably have accepted that rule.

I seek to understand and adjudicate Wiredu's challenge to Lewis as much as possible on its own terms. So I take Lewis's antecedent commitments, as Wiredu sees them and as much as possible, as given. First, like Wiredu, I do not dispute or attempt to fully articulate Lewis's pragmatism about choice of logical system. Lewis said that "the grounds of choice [among alternative logical systems] can only be pragmatic", "such as simplicity or comprehensiveness or accord with our most frequent purposes in inference" (Lewis 1932, 507; 1934, 74). Though Lewis gave "little theoretical discussion of these 'pragmatic considerations"' (Parry 1968, 153), I do not discourage any inclination readers may have to equate Lewis's 'pragmatic grounds' with 'epistemic justification'. Granted, "unlike Wilfrid Sellars and many others in the latter half of the 20th century, Lewis never recognized such [pragmatic] factors as criteria of empirical [epistemic] justification" (Hunter 2016, §4). Nevertheless, Lewis did not dispute William Parry's account of Lewis's pragmatism about choice of logical system, as being concerned with a "question of *truth*" about our ordinary concept of logical implication (Parry 1968, 154) and as including intuitions about logical and modal concepts among 'pragmatic grounds' in Lewis's sense (Parry 1968, 126; Lewis 1968, 658; see also Cresswell 1967). (Obviously, equating Lewis's 'pragmatic grounds' with 'epistemic justification' would not commit Lewis to reducing either necessity itself or the meaning of 'necessary' to epistemic terms.) Second, I do not dispute either Wiredu's and Lewis's shared (but not universally shared) intuitions about logical implication and about modality, or their shared (but problematic) acceptance of the view that "Strict implication is identical with deducibility", that is, with logical implication (Wiredu 1973, 50; see Lewis and Langford 1959, ch. 8). Finally, I use 'S has justification to accept p' (and analogously for rejection of or suspension of judgment about p) to abbreviate 'S has some prima facie, pragmatic grounds (in Lewis's sense) to accept p', where the latter is understood not to imply either that p is true, or that S is aware of having those grounds, or that S does not also have some contrary grounds, or that S does accept p.

In section 2, I reconstruct Wiredu's primary argument and show that it cannot be accepted as sound. In section 3, I amend Wiredu's argument and show that the initial variant argument I construct can be accepted as sound. In section 4, I further amend Wiredu's argument and further address relations among Wiredu's primary and alternative arguments, the final variant argument I construct, Makinson's and Humberstone's notes, and Cresswell's argument.

2 Wiredu's primary argument

In this section I reconstruct Wiredu's primary argument and show that it cannot be accepted as sound, since two essential premises in the argument have indeterminate contents and Wiredu gives inadequate reasons for those premises.

Wiredu's argument turns on the two special modal systems he constructs as extensions of S1 and S2, which I call W1 and W2, and on his special "strong rule of substitution" (Wiredu 1979, 691), which I call WR. Here is Wiredu's primary argument as I see it.

Wiredu's primary argument

(P1) Lewis had justification to accept S2. (He was "cautious of any system stronger than S2".)

(P2) Lewis had justification to accept WR. (That rule is "very plausible and intuitively acceptable".)

(P3) S4 is deductively contained in W1 and in W2. (S4 "becomes available".)
(P4) Justification is relevantly transmitted by deduction. (Implicit.)
(C5) So, Lewis had justification to accept S4. (He "could have been more adventurous modally [for] he was committed to S4".)

The circumstantial *ad hominem* nature of Wiredu's argument is clear from the way its premises and conclusion are centered on Lewis's personal pragmatic (or, if readers prefer, epistemic) circumstances. The conclusion, for example, is not 'the truth about logical necessity includes S4' but 'Lewis had justification to accept that the truth about logical necessity includes S4'. For charity to Wiredu, 'had justification' here means 'had some prima facie grounds', not 'had overriding or all-in grounds'. Since Wiredu does not discuss Lewis's objections to systems stronger than S2, my reading of the argument leaves it as a further case to be made that the grounds provided by WR to Lewis to accept S4 would have been sufficiently strong to trump those objections.

Premises P1 and P4 can be granted for present purposes. P1 reflects Wiredu's decision to take Lewis's antecedent commitments as given. P4 ensures that the argument's conclusion in this case validly follows from the first three premises jointly, but does not imply that justification in general is deductively closed.

Premises P2 and P3, on the other hand, cannot simply be granted. On the one hand, since S4 is not deductively contained in S2, Wiredu's conclusion C5 does not follow from P1 and P4 alone, and P2 and P3 are essential to his argument. On the other hand, it is not clear what P2 and P3 amount to, or why they should be thought to be true.

P2's content is indeterminate since WR is indeterminate. Adopting Lewis's notation of $=$ for strict equivalence (mutual strict implication) (Wiredu 1979, 689), Wiredu states his "strong rule of substitution" as follows.

(WR) "*If* $(p = q)$, *then* $[A(p) = A(q/p)]$ (Here...$A(q/p)$ is like $A(p)$ except for containing q in place of p in all or some of the occurrences of p in A)" (Wiredu 1979, 691).

The difficulty is how to understand this. Wiredu says he intends WR as a "strengthening of the rule for the substitution of strict equivalents", that is, "the rule that is standardly used in the formulation of S1 and of the Lewis systems, generally" (Wiredu 1979, 692, 691). But he does not say exactly what strengthening measure he has in mind from among indefinitely many conceivable sorts. In another paper he acknowledges the convention of suppressing turnstiles in statements of inference rules (Wiredu 1973, 39). But if WR were read in light of that convention, then WR would turn out just to be the standard rule for substitution of proved strict equivalents (Eq′), already available in S1 and the Lewis systems generally, which Wiredu insists is distinct from his rule.

(Eq′) If $\vdash (\gamma = \delta)$, then $\vdash (\alpha = \beta)$, provided that α differs from β only in having γ in some of the places where β has δ. (Hughes and Cresswell 1996 [hereafter NIML], 199–200; Hughes and Cresswell 1968 [hereafter IML], 246–247; Wiredu 1979, 690).

Wiredu invokes WR in his proofs only once, to license a single step in his ostensible derivation of "the characteristic thesis of S4" (4.1) in W1 and W2 (Wiredu 1979, 690).

(4.1) $\Box p \prec \Box\Box p$

Wiredu evidently intends some sort of weakening or other of the restrictions on Eq′, sufficient for WR to license that one proof-step. More than that is not easy to say.

WR's indeterminateness also infects the reasons Wiredu offers for P2. On the one hand, Wiredu's assertion that WR is "very plausible and intuitively acceptable" is empty if he actually does not have any determinate rule in mind. On the other hand, though the "very informal" subsidiary argument Wiredu offers for P2 (Wiredu 1979, 691–692) is too terse and opaque for me to confidently reconstruct or remark upon, that subsidiary argument does seem to involve three elements available in S4 but not available to Wiredu on pain of circularity of justification. First, proved material equivalents seem to be taken as everywhere substitutable, though that rule (Eq) is not available in S1, S2, or S3 (IML, 228).

(Eq) If $\vdash (\gamma \equiv \delta)$, then $\vdash (\alpha \equiv \beta)$, provided that α differs from β only in having γ in some of the places where β has δ. (NIML, 32; IML, 35)

Second, necessity seems to be taken as everywhere inferable from provability, though that rule (N) is not available in S1, S2, or S3 (IML, 235).

(N) If $\vdash \alpha$, then $\vdash \Box\alpha$. (NIML, 361; IML, 346)

Third, strict equivalence seems to be taken as substitutably equivalent to necessary material equivalence, though that thesis (2.1) is not available in S1 (Feys 1965, 45, 62, 72).

(2.1) $(p = q) = \Box(p \equiv q)$

If the subsidiary argument does involve elements available in S4 but not in S1 or S2, then, regardless of however WR might be rendered determinate, it is difficult to see how the subsidiary argument can render WR "very plausible and intuitively acceptable" in a way or to a degree that 4.1 was not already plausible and acceptable to Lewis.

P3's content is also infected by WR's indeterminateness, since W1 and W2 are defined in terms of WR along with specific definitions of strict equivalence (D=) and possibility (D$\Diamond$) (Wiredu 1979, 689).

(D=) $p = q \ =_{df} \ [(p \prec q) \ \& \ (q \prec p)]$
(D$\Diamond$) $\Diamond p =_{df} \sim [(p = (r \ \& \ \sim r)]$
(W1) W1 is S1 with D= and extended with WR as an additional primitive inference rule and D$\Diamond$ as an additional definition, and with the extended system closed also under S1's primitive inference rules.
(W2) W2 is S2 with D= and extended with WR as an additional primitive inference rule, and with the extended system closed also under S2's primitive inference rules.

(W2's definition omits D$\Diamond$ because S2 has equivalent thesis 2.2. See Wiredu 1979, 692; Lewis and Langford 1959, 506.)

(2.2) $\sim \Diamond p = \ [(p = (r \ \& \ \sim r)]$

So, W1 and W2 are indeterminate because WR is indeterminate. (D= and D$\Diamond$ are also problematic for W1 and W2, since Lewis's bases for S1 and S2 seem to have both strict equivalence and possibility as "primitive or undefined ideas". See Lewis and Langford 1959, 123; but compare IML, 217. No 'idea' can be at once both undefined and defined in any basis for a system. See Parry 1968, 129, note 51. But Wiredu does not specify non-Lewis bases for S1 and S2 in W1 and W2.)

Besides the indeterminateness of P3's content, Wiredu gives faulty proofs for his claim that extending either S1 to W1 or S2 to W2 makes 4.1 provable and thus yields at least S4. Wiredu repeatedly but mistakenly says that, except for WR and D$\Diamond$ (or 2.2), his proofs use "only principles available in S1" (Wiredu 1979, 691; see also 690). The "technical error" reported by Makinson is that Wiredu twice relies on the rule of conditional proof for strict implication, which is not a valid rule in S1, S2, or S3. If that rule were valid in any of those systems, then replacement of $\supset$ with $\prec$ as the main operator in a thesis would always yield another thesis. But each of those systems has some "theses in which the main operator is $\supset$ but which cease to be theses if the $\supset$ is replaced by $\prec$" (IML, 228). Another error, not reported by Makinson, is that Wiredu also relies on a "simple lemma" (L0) that he mistakenly claims is valid in S1 (hence in S2 and S3) (Wiredu 1979, 690).

(L0) $\sim (p = p) = \bot$

L0 fails, for example, left-to-right in the S3 (hence S2 and S1) model where $W = \{w_1, w_2\}$, $N = \{w_1\}$, $Q = \{w_2\}$, $R = \{\langle w_1, w_1\rangle, \langle w_1, w_2\rangle\}$, and $V(p, w_1) = V(p, w_2) = 1$. (Here $=$ is identity. Kripke models for non-normal systems distinguish between normal and non-normal worlds. See IML, 274–276; NIML, 201–202. Here N comprises the normal worlds in W, while Q comprises the non-normal worlds in W.) All these faults in Wiredu's proofs are independent of WR's indeterminateness and independent of WR's role in the proofs. So, Wiredu's proofs fail on account of these faults regardless of however WR (hence also W1 and W2) might be rendered determinate and regardless of whether the single proof-step that Wiredu intends WR to license is a valid proof-step or not. Wiredu's faulty proofs thus do not provide any reason to accept that S4 is deductively contained in W1 and in W2.

In sum, Wiredu's primary argument cannot be accepted as sound, since P2 and P3, while essential to the argument, have indeterminate contents and Wiredu gives inadequate reasons for those premises.

3 Initial variant argument

In this section I amend Wiredu's primary argument in a number of specific ways that jointly change the argument's view of Lewis's relevant circumstances (pragmatic or epistemic, as readers prefer). The resulting initial variant of Wiredu's argument can be accepted as sound.

First, I replace the weak non-normal systems S1 and S2 in P1, P2, and P3 with the only slightly stronger normal system T (not mentioned by Wiredu) that contains both S1 and S2 and like them is contained in S4. This amendment reflects Cresswell's suggestion that, rather than S1 or S2, "the system most closely satisfying Lewis's intuitions would be T" (Cresswell 1967, 204). The suggestion in other words is that, Lewis had grounds to accept T even though he did not accept T. Now, T lacks 4.1 but has 2.1, 2.2, Eq, Eq′, N, L0, conditional proof for strict implication, and 1.1.

(1.1) $(p = q) \;=\; [(p \mathrel{-\!\!3} q) \;\&\; (q \mathrel{-\!\!3} p)]$

So, T is weaker than S4 but shares crucial features of S2 with Wiredu's W1 and W2, and also has non-S2 principles whose intuitive attractiveness for Wiredu seems likely responsible for his failure to see the formal faults in his own proofs and arguments.

Second, I replace Wiredu's indeterminate rule WR in P2 and P3 with a determinate rule that strengthens Eq′ in a precise way to permit substitution, not only of proved strict equivalents (whose substitution Eq′ permits), but also of strict equivalents that are logically implied by (deducible from) assumptions or premises that are not themselves provable. In connection with Wiredu's paper, Humberstone uses a rule he calls ($\Box \leftrightarrow$ E), but which for present purposes I rewrite as MSR.

(MSR) If $\vdash (\chi \supset \Box(\gamma \equiv \delta))$, then $\vdash (\chi \supset (\alpha \supset \beta))$, provided that α differs from β only in having γ in some of the places where β has δ. (See Humberstone 2011, 604; compare Cresswell 1965, 192, Axiom Schema I and Def =. I follow Wiredu in using $\supset$ for material implication, $\equiv$ for material equivalence, and no symbol for propositional identity, while Humberstone uses $\rightarrow$ for material implication, $\leftrightarrow$ for material equivalence, and $\equiv$ for propositional identity.)

But, at least in his primary argument, Wiredu frames WR in terms of strict equivalence (mutual strict implication), not in terms of necessary material equivalence. In another context and without mentioning Wiredu, Steffen Lewitzka strengthens Eq′ to a rule he calls SR, which for present purposes I rewrite as follows.

(SR) If $\vdash (\chi \supset (\gamma = \delta))$, then $\vdash (\chi \supset (\alpha = \beta))$, provided that α differs from β only in having γ in some of the places where β has δ. (See Lewitzka 2016, 1771. Lewitzka uses $\rightarrow$ for material implication, $\leftrightarrow$ for material equivalence, and $\equiv$ for both strict equivalence and propositional identity.)

But Wiredu follows Lewis in identifying logical implication (deducibility) with strict implication, not material implication (even if proved). For sake of least divergence from Wiredu's text, I prefer to amend his primary argument by replacing WR, not with MSR or SR, but with the rule I call □SR.

(□SR) If $\vdash (\chi \dashv\!\!\!\!\!3\ (\gamma = \delta))$, then $\vdash (\chi \dashv\!\!\!\!\!3\ (\alpha = \beta))$, provided that α differs from β only in having γ in some of the places where β has δ. (See Martens 2019. Though Lewitzka 2016, 1774, describes an equivalent axiom schema, which he calls □SP, he does not describe □SR.)

□SR permits substitution, not only of proved strict equivalents, but, more generally, of strict equivalents that are formally deducible by Wiredu's lights. Replacing WR with □SR in Wiredu's proof of 4.1 renders the amended proof valid in T. The amended proof thus establishes that S4 is deductively contained in T+□SR. (S4 has □SR, but T lacks □SR. See Martens 2019.)

Third, I add to P1 the assumption that Lewis had some prima facie grounds to accept the problematic view that strict equivalence is propositional identity, or, in other words, that to say '*p* and *q* are strictly equivalent' is to say '*p* and *q* are the same proposition'. (For favorable treatments of the view, see Stalnaker 1984 and Cresswell 1965; for criticisms, see Prior 1963 and Cresswell 1988.) I think Lewis's exact position on this view is not easy to pin down and there is no space here for full discussion of the interpretive issues. What is important for present purposes is that Lewis arguably took the view seriously and often wrote in way that invites attribution of the view to him. For example, he repeatedly said of various statements of strict equivalence that they assert "identity" of propositions (Lewis and Langford 1959, 128, 129). As for Wiredu, though he does not mention propositional identity, I am tempted to conjecture that his engagement with Lewis has encouraged him to tacitly and unreflectively see Lewis as identifying strict equivalence with propositional identity, and to tacitly and unreflectively make that identification himself. (See IML, 347, on "the chance of confusion" when = is used for strict equivalence.)

Fourth, I further amend P2 to reflect the fact that, since □SR is an unqualified form or analogue of Leibniz's Law ('identicals are indiscernible') for propositional identity if strict equivalence is propositional identity, the intuitive plausibility of that Law provides anyone who has grounds to accept the latter view with grounds also to accept □SR. (Prior 1963, §1, discusses propositional identity in relation to Leibniz's Law and substitution of equivalent propositions. Cresswell 1965, 195, constrains propositional identity with an unqualified form or analogue of Leibniz's Law that he calls "the identity schema". Humberstone 2011, 604, 603, calls $(\Box \leftrightarrow \mathrm{E})$ "the analogous 'strict equivalence' version of $(\equiv \mathrm{E})$", where the latter is an unqualified form or analogue of Leibniz's Law for propositional identity. Lewitzka 2016, 1771, associates his strengthened substitution rule SR with "a general ontological law known as *the Indiscernibility of Identicals*".) However, since T lacks □SR despite having Eq, N, and 2.1, the "very informal" subsidiary argument Wiredu offers to justify WR cannot be rehabilitated to provide additional grounds for □SR beyond the latter's intuitive plausibility as an unqualified form or analogue of Leibniz's Law.

Fifth, I amend the argument's form to make it conditional, since P1 as thus far amended raises interpretive issues for which there is no space to discuss fully here. For example, though Lewis "does not mention T" (Cresswell 1967, 204, note 18), the fact that he did not accept T but stood fast at S2 "to the last" (as Wiredu says), suggests that either he thought he did not have justification to accept T (contrary to Cresswell's suggestion), or he thought he had stronger justification not to accept T. Moreover, though above I cited some places where Lewis seems to have accepted that strict equivalence is propositional identity, other places can be cited where he seems to have rejected that view (for example, Lewis and Langford 1959, 285, note 2). To set aside interpretive issues raised by P1 but without amending that argument-element's content, I change its function from asserted premise to assumption for conditional proof and accordingly change its designation to A1. I discharge the dependence on A1 of now-intermediate conclusion C5 by conditional proof to a new final conclusion C6.

Here now is my initial variant of Wiredu's primary argument.

Initial variant argument

(A1) Lewis had justification to accept T and he had justification to accept that strict equivalence is propositional identity. (Assumption for conditional proof.)

(P2) If Lewis had justification to accept that strict equivalence is propositional identity, then he had justification to accept □SR. (□SR is an unqualified form or analogue of Leibniz's Law for propositional identity if strict equivalence is propositional identity.)

(P3) S4 is deductively contained in T+□SR. (By amendment of Wiredu's 'primary argument' proof.)

(P4) Justification is relevantly transmitted by deduction. (Granted to Wiredu.)

(C5) So, Lewis had justification to accept S4. (From A1, P2, and P3, by P4.)

(C6) So, if Lewis had justification to accept T and he had justification to accept that strict equivalence is propositional identity, then he had justification to accept S4. (From A1 and C5, by conditional proof, but retaining C5's dependence on P2, P3, and P4.)

The variant argument retains the circumstantial *ad hominem* nature of Wiredu's argument, for the premises and conclusions remain centered on Lewis's personal pragmatic (or epistemic) circumstances. Also as before, 'had justification' here still means 'had some prima facie grounds', not 'had overriding or all-in grounds'. Since the variant argument still does not address Lewis's objections to systems stronger than S2, a further case still remains to be made that the grounds provided, by □SR and by the view that strict equivalence is propositional identity, to Lewis to accept S4 would have had sufficient collective strength to trump those objections. The variant argument thus leaves open the possibility that Lewis could justifiably have resisted accepting S4, for example, by justifiably rejecting the view that strict equivalence is propositional identity or by justifiably accepting Leibniz's Law for propositional identity only in a qualified form. (Systems for propositional identity with necessity weaker than S4 are investigated, for example, by Ishii 1998; 2000; Martens 2004; and Lewitzka 2015; 2016.)

Since the initial variant argument is valid and its asserted premises (P2, P3, P4) all can reasonably be accepted as true, the initial variant argument can reasonably be accepted as sound.

4 Final variant argument

In this section I further amend Wiredu's argument and further address relations among Wiredu's primary and alternative arguments, the final variant argument I construct, Makinson's and Humberstone's notes, and Cresswell's argument.

In section 2 above I reconstructed Wiredu's primary argument, confirmed Makinson's report of "a technical error" in it, and showed that it cannot be accepted as sound. In section 3, reluctant to accept Makinson's assessment that its problematic character "vitiates" Wiredu's primary argument, I noted Cresswell's suggestion that Lewis's explicit expressions might not always accurately reflect his motivating intuitions, applied that suggestion with similar charity to Wiredu's expression of his primary argument, and amended the argument to yield an initial variant argument that can be accepted as sound. I now show how Wiredu's primary and alternative arguments can be understood as providing two formally distinct lines of support for a key premise in a single overall argument for Wiredu's main conclusion.

The alternative argument is aimed at showing that "the addition of [a certain] definition of necessity [...] to a suitable system of propositional logic yields at least S4" (Wiredu 1979, 693). On Makinson's report, the same technical error that "vitiates" the primary argument also "vitiates" the alternative argument. Makinson's reason for saying this clearly is Wiredu's statement in justification of a crucial step in the alternative argument that "The proof of this is essentially the same as the one already given for [4.1 in the primary argument]" (Wiredu 1979, 693), that is, the one with a technical error. But Makinson clearly has been hasty, for Wiredu immediately completes his sentence with "except that here the conclusion is a material implication instead of a strict implication". Since the conclusion here is by conditional proof for material implication, not for strict implication, one instance of the technical error, at least, has not carried over from the primary argument to the alternative argument. On the other hand, Makinson also reports "some confusion" in the alternative argument and it is not immediately clear how that charge might be avoided. Wiredu does not say anything to suggest that the proof carried over from the primary argument to the alternative argument is not still intended to rely on L0 and WR. But he does say that in the alternative argument = is no longer strict equivalence (mutual strict implication). He says, "In the [alternative argument] we understand $p = q$ as an abbreviation for '$p \equiv q$ is logically true (i.e., tautological)' " (Wiredu 1979, 693). On the one hand, it is not immediately clear how nested occurrences of =, as in L0, should now be understood. On the other hand, the shift in WR's content between the primary argument and the secondary argument aggravates suspicion that WR is indeterminate.

Faced with my ambivalent assessment of Makinson's report on Wiredu's alternative argument, I defer to Humberstone. It seems to me that Humberstone's attention is focused on formal aspects of the alternative argument, which he takes to be in relevant respects essentially the same as Cresswell's 1965 argument, and which he thinks can be understood in such a way as to be sound and free from any insuperable technical error or confusion of the sort Makinson reports. I take Humberstone's report to be that, with suitable amendments in either case, Wiredu's alternative argument and Cresswell's argument both establish that S4 results when the smallest normal system K (not mentioned by Wiredu) is extended with MSR.

Here now is my final variant of Wiredu's overall argument, which incorporates his primary and alternative arguments at P3a and P3b below.

Final variant argument

(A1) Lewis had justification to accept T and he had justification to accept that strict equivalence is propositional identity. (Assumption for conditional proof.)

(P2) If Lewis had justification to accept T and he had justification to accept that strict equivalence is propositional identity, then he had justification to accept K and he had justification to accept □SR and he had justification to accept MSR. (K is deductively contained in T, and □SR is an unqualified form or analogue of Leibniz's Law for propositional identity if strict equivalence is propositional identity, as is MSR, given that T has 2.1 and Eq′.)

(P3a) S4 is deductively contained in T+□SR. (By amendment of Wiredu's 'primary argument' proof.)

(P3b) S4 is deductively contained in K+MSR. (By amendment of Wiredu's 'alternative argument' proof or that of Cresswell 1965. See Humberstone 2011, 604.)

(P4) Justification is relevantly transmitted by deduction. (Granted to Wiredu.)

(C5) So, Lewis had justification to accept S4. (By P4, from A1, P2, and either P3a or P3b.)

(C6) So, if Lewis had justification to accept T and he had justification to accept that strict equivalence is propositional identity, then he had justification to accept S4. (From A1 and C5, by conditional proof, but retaining C5's dependence on P2, P4, and either P3a or P3b.)

The final variant argument retains the circumstantial *ad hominem* nature of Wiredu's argument and 'had justification' here still means 'had some prima facie grounds', not 'had overriding or all-in grounds'. Since the final variant argument is valid and its asserted premises (P2, P3a, P3b, P4) all can reasonably be accepted as true, the final variant argument can reasonably be accepted as sound.

The final variant argument is not exactly what Wiredu says but it is what he means, I think, in the sense that it accurately expresses the interesting philosophical intuitions motivating him. The argument also serves as a model of the Lewisian pragmatic approach to the question of choice of modal logic. On that approach, the question is never settled by formal methods but only by explicit recourse to specific pragmatic factors rooted in specific circumstances. There is some irony in Wiredu's teasing of Lewis for being "cautious" rather than "adventurous modally", when Wiredu himself is so philosophically careful and circumspect in his respectful treatment of Lewis's pragmatic circumstances.

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RESEARCH ARTICLE

On 'actually' and 'dthat': Truth-conditional Differences in Possible Worlds Semantics

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Abstract: Although possible worlds semantics is a powerful tool to represent the semantic properties of natural language sentences, it has been often argued that it is too coarse: with the tools that possible worlds semantics puts at our disposal, any relevant semantic difference has to be a truth conditional difference representable as a difference in intension. A case that raises questions about the ability of possible worlds semantics to make the appropriate discriminations is the distinction between rigidity and direct reference, an issue deeply connected to the representation of the behaviour of two operators: 'dthat' and 'actually'. Differences between the mode of operation of 'dthat' and 'actually' have been observed, but they have not been examined in depth. Our purpose is to explore systematically to what extent the observed differences between the two operators have truth conditional consequences that are formally representable in possible worlds semantics.

Keywords: Actuality operator; direct reference; dthat; possible worlds semantics; rigidity

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1 Possible worlds semantics

Semantics is often characterized as a theory of truth conditions. Differences in meaning between two expressions are never as clearly demonstrated as when it is possible to point out differences in the truth conditions of the sentences in which the expressions in question figure and, in spite of the emergence of other systems and approaches, possible worlds semantics continues to be the most powerful tool of formal representation in semantics, affording a precise characterization of the truth conditions of sentences as intensions, i.e., functions from possible worlds to truth values. In general, the representation of meanings in terms of functions from possible worlds to the appropriate kind of extensions delivers a compositional semantics that formally captures the semantic operation of non-syncategorematic expressions in an ordinary language.

Although possible worlds semantics is clearly a powerful tool to represent faithfully the semantic properties of natural language sentences, it has been often argued that it is too coarse. There are important properties of sentences that are not discriminated in possible worlds semantics. One interesting case that raises questions about the ability of possible worlds semantics to make the appropriate discriminations is the distinction between rigidity and direct reference, an issue deeply connected to the representation of the behaviour of two operators: 'dthat' and 'actually'. Some differences between the mode of operation of 'dthat' and 'actually' have been pointed out by several authors.[1] Our purpose here is to explore formally how different these operators really are and to what extent the observed differences do have truth conditional consequences representable in possible world semantics.

2 Rigidity and direct reference

Kripke's notion of rigidity, a notion that underpins one of the most important revolutions in semantic theory, was presented within the framework of possible worlds semantics: a rigid designator designates the same object in every possible world. Hence, 'the successor of 7' is rigid because it designates the number 8 in every possible world, whereas 'the tutor of Alexander the Great' is not

[1]Most notably by (Salmon 1991). See also (Soames 2005, 28-30) for a summary.

rigid, since someone different from Aristotle could have tutored Alexander.[2] Kripke’s revolutionary claim was that names are rigid designators, whereas most definite descriptions, in particular the descriptions that were associated with names according to classical descriptivists, are not.

It has been typically assumed that a rigid designator designates nothing in a world in which the actual designatum fails to exist, following (Kripke 1971, 146) and (Kripke 1980, 49): “a designator rigidly designates a certain object if it designates that object wherever the object exists.” But this has been the source of some controversy and even Kripke has suggested that some rigid designators, names in particular, designate even in worlds in which the designatum fails to exist (see (Kaplan 1989, 570, fn. 8), suggesting that names are not just rigid but *obstinately* rigid.[3] This is an important issue, and we will return to it.

A non-rigid designator can be rigidified using the intensional operator ‘actually’ or ‘actual’. ‘Actually’ operates on formulas: the interpretation of *actually* ϕ at any possible world is the interpretation of ϕ at the actual world. When combined with definite descriptions, *the* x : *actually* Px (or, closer to natural language, *the actual* P) designates i in every possible world in which i exists, if *the* x : Px designates i in the actual world. For instance, ‘the actual tutor of Alexander the Great’ designates Aristotle in every world w in which Aristotle exists, for Aristotle is the individual in w that satisfies being tutor of Alexander in the actual world.

Direct reference, a different revolutionary idea introduced by Kaplan, appeared on the scene roughly around the time Kripke introduced his distinction between rigid and non-rigid designators. Unlike rigidity, direct reference relies on the Russellian picture of structured propositions. A term is directly referential just in case it contributes its referent to the proposition expressed by sentences containing it. Propositions are represented by Kaplan as n-tuples that contain the elements contributed to truth-conditional content by expressions in sentences. Hence, in (Kaplan 1978) we learn that the truth-conditional content, or proposition expressed by a sentence of the form n *is* Q, where n is directly referential, is singular, a proposition of the form $< i, Q^* >$, where i is the referent of n and Q^* is the property expressed by the predicate Q. In contrast, the propositional contribution of a non-directly referential term,

[2]This, of course, takes for granted that teaching Alexander was not an essential property of Aristotle, a plausible assumption.

[3]See (LaPorte 2018, section 1.2).

a definite description such as *the* P, is general, an attributive complex that selects at each index of evaluation w the individual relevant for the computation of truth value at w. Hence, *the* P *is* Q expresses a proposition that can be represented as $<< \text{‘}the\text{’}, P^* >, Q^* >$. The difference between singular propositions, those expressed by sentences containing directly referential terms, and general propositions can be best captured as a difference between the constituents of truth conditional content: in the case of a singular proposition, a proposition of the form $< i, Q^* >$, it is i itself that determines truth value at each index of evaluation, whereas in the case of a general proposition, a proposition of the form $<< \text{‘}the\text{’}, P^* >, Q^* >$ an attributive complex (constrained by a condition of uniqueness) selects for each index the individual that will determine truth value at that index. Even if the attributive complex $< \text{‘}the\text{’}, P^* >$ selects i at each index, the truth conditional content, i.e., that which determines truth conditions, is different in each case. Or, in other words, even if *the* P is a rigid definite description that designates i, the items determining truth conditions that correspond to *the* P and to n are quite different.

Names, indexicals and demonstratives are, according to Kaplan, paradigmatic instances of directly referential terms. But Kaplan also introduces a device that operates on definite descriptions and produces directly referential terms: the ‘dthat’ operator. The idea behind the mode of operation of ‘dthat’ can be captured if one thinks of a definite description as a demonstration, like a pointing. When a speaker ostensibly points at a person i and utters ‘that P is Q’, i becomes the truth conditional content of the sentence uttered. In a similar vein, if *the* P designates i, $dthat(the\, P)$ contributes i, not the complex $< \text{‘}the\text{’}, P^* >$ to the truth conditional content. Observe, however, that the contribution of *the actual* P to content, to put it in Kaplanian terms, is not i, but rather an attributive complex corresponding to *the actual* P. Hence, the mode of operation of ‘dthat’ is quite different from the mode of operation of ‘actual’ or ‘actually’. But the important question for our purposes is whether that difference generates a difference in possible-world-representable truth conditions.

3 The semantic operation of ‘actually’ and ‘dthat’

The combination of the actuality operator (A) with definite descriptions raises an interesting issue as regards the index at which the requirement of uniqueness should be satisfied. We have two prima facie plausible choices: the world of evaluation, or the actual world. Let w be a possible world, $D(w)$ the domain of individuals existing at w, @ the actual world and $Int(\phi)^w$ the interpretation of an expression ϕ at w.[4] The choices can be represented as follows:

(a) $Int(the\,x : APx)^w = i$ if $i \in D(w)$ and i is the unique individual in $D(w)$ that satisfies Px in @. $Int(the\,x : APx)^w$ is undefined otherwise.

(b) $Int(the\,x : APx)^w = i$ if $i \in D(w)$ and i is the unique individual in $D(@)$ that satisfies Px in @. $Int(the\,x : APx)^w$ is undefined otherwise.

If we put it in Russellian terms, the question is which one of these two sentences (corresponding to (a) and (b)) best captures the logical form of *the actual P is Q*, a difference that hinges on whether the uniqueness condition falls under the scope of ‘actually’:

(a’) $\exists x[APx \wedge \forall y(APy \rightarrow x = y) \wedge Qx]$

(b’) $\exists x[A(Px \wedge \forall y(Py \rightarrow x = y)) \wedge Qx]$

In (a’) we are treating *actually* P as a predicate: there is an i in w such that i is actually P and anything in w that is actually P is that very i (and i is Q).

In (b’) the condition of uniquely satisfying P is meant to be satisfied in the actual world: there is an i in w such that actually i is the only P (and i is Q).

Both are possible interpretations, and many informal discussions of the role of rigidified definite descriptions fail to distinguish between the two. This is because, typically, illustrations of the behavior of ‘actual’ use definite descriptions such as ‘the actual President of the US’ a description that designates Trump in a world w in which both Trump and Clinton exist and Clinton wins the 2016 election. Since there is only one President of the US in @ and, by assumption, also in w, the question as regards the locus of satisfaction of the uniqueness requirement is not addressed.

[4]The interpretation funtion Int is defined relative to a structure, an index of evaluation and an assignment, but we are abbreviating the technical details, which are trivial.

In our view, there is a clear motivation to go for (a/a') rather than (b/b'). Consider the term 'the only actually existing person'. In a world in which Pat is the only one of the actual human beings that exists, surrounded by human possibilia, it should be true that Pat is the only actually existing human being and hence that 'the only actually existing person' designates her relative to that world.

Yet, Pat will not uniquely satisfy *Exists x* in @, and so the description in question will be denotationless in w if (b) is the interpretation of choice. Or, if (b') is the chosen representation, any sentence in which the description figures will be false:

$\exists x[A(Exists\, x \wedge \forall y(Exists\, y \rightarrow x = y)) \wedge Qx]$

for the uniqueness condition under the scope of the operator A will not be satisfied if there are in the actual world at least two people.[5]

Hence, we will take (a) and (a') to be the correct choices for the interpretation of sentences in which descriptions of the form the actual P occur.[6]

On the other hand, the interpretation of $dthat(the\, P)$ is given by the following clause:

$Int(dthat(the\, P))^w = i$ if $i \in D(@)$ and i is the unique individual in $D(@)$ that satisfies Px in @. $Int(dthat(the\, P))^w$ is undefined otherwise.

We can try to capture the logical form of sentences containing the operator 'dthat' in terms of quantifiers, taking inspiration from Russell. For a sentence such as $dthat(the\, P)$ *is* Q, the formalizations (a') or (b') will not do. When we evaluate $dthat(the\, P)$ *is* Q in any world w, we need to select an individual in the domain of the actual world, and the existential quantifier, both in (a') and (b'), search for individuals in w. Adding an actuality operator in front of the existential quantifier in (a') or (b') will not do, since then the truth of the sentence will depend on whether the individual in @ which is uniquely P in @ is Q in @, and what we really need is an individual which is uniquely P in @ and Q in w. Hence we need to be able to quantify over the domain of the actual world even when we are evaluating in a different world w.

The actuality quantifiers ($\forall^@, \exists^@$), introduced by Allen Hazen (1990), are

[5]Of course, given that we identify the domain of a world with what exists at that world, that formula is just equivalent to $\exists x[A\forall y(x = y) \wedge Qx]$.

[6](Soames 2005, 30, fn. 22), reporting a suggestion by Ali Kazmi, proposes that what we have called (b) is a "highly intuitive reading [of sentences of the form *the actual P is Q*] in which the uniqueness condition is correctly imposed on things that have the property expressed by [P] in the world-state of the context [namely, the actual world]". Soames gives no indication why he regards this reading as highly intuitive but, as we think our example involving 'the actually existing person' shows, that reading is not adequate.

designed precisely to do that.[7] In general,

$Int(\forall^{@}x\phi(x))^{w} = true$ iff for all $i \in D(@)$, $Int(\phi(x)[i/x])^{w} = true$ (analogously for $\exists^{@}$).

With the help of those quantifiers, we can represent the sentence *dthat*(*the P*) *is Q* as:

(a*) $\exists^{@}x[APx \wedge \forall^{@}y(APy \rightarrow x = y) \wedge Qx]$

We could also use a formalization similar to (b’), namely

(b*) $\exists^{@}x[A(Px \wedge \forall^{@}y(Py \rightarrow x = y)) \wedge Qx]$,

but in this case (a*) and (b*) are equivalent. The difference between (a’) and (b’) had to do with the locus of satisfaction of the uniqueness condition. In the case of ‘dthat’ the uniqueness condition has to be satisfied in the actual world, otherwise *dthat*(*the P*) is denotationless with respect to all indices. (b*) explicitly captures that condition by including the uniqueness condition under the scope of @. But (a*) achieves the same result, by restricting the domain of the universal quantifier to the actual world.

4 A merely conceptual difference between ‘dthat’ and ‘actually’?

Clearly there is an important conceptual difference between ‘dthat’ and the ‘actually’ operator. Resorting again to the picture of structured propositions, whereas *dthat*(*the P*) *is Q* expresses a singular proposition of the form $< i, Q* >$, *the actual P is Q* expresses a general proposition of the form $<< \text{‘}the\text{’}, Actual\, P* >, Q* >$ or, in other words, whereas in the case of *dthat*(*the P*) *is Q*, i is the object that is provided as the determiner of truth value at each index, in the case of *the actual P is Q* an attributive complex selects, among the individuals in the domain of any given world w, the unique individual that happens to be P in the actual world.

To give an example, let us suppose for the moment that Aristotle exists in all worlds: ‘the actual tutor of Alexander the Great was a philosopher’

[7] Hazen actuality quantifiers should not be confused with the ordinary quantifiers (usually misnomed ‘actualist quantifiers’, in order to distinguish them from possibilist quantifiers). As Hazen notes, ordinary quantifiers are really world-restricted, since they always range over the domain of evaluation. Hazen’s aim is different from ours: he shows that the actuality quantifiers and the operator ‘actually’ both expand the expressive power of first-order logic, and he also shows that both devices are not equivalent. Moreover, Hazen’s language does not incorporate definite descriptions.

expresses a general proposition whose truth value at an index w depends on whether the unique individual in the domain of w that happens to be actually tutor of Alexander (namely, Aristotle) is a philosopher in w. 'Dthat (the tutor of Alexander the Great) was a philosopher', on the other hand, expresses a singular proposition whose truth value at each index depends on whether Aristotle is a philosopher at that index.

This is all well, and one can see that, conceptually, there is an important difference here in how the truth conditions of the respective sentences are captured and in how truth value at an index is determined. But the difference affects the *how* not the *what.* Direct reference proponents may insist on the importance of the distinction between the two kinds of content, but the fact is that no truth conditional difference between the two sentences is manifested: the functions from possible worlds to truth values that represent their truth conditions are the same. As (Soames 2005, 28n.) puts it, "this difference between dthat-rigidified descriptions and actually-rigidified descriptions [...] all but washes away in semantic systems in which the content of an expression in a context is identified with its intension."

That there is no truth conditional difference between 'dthat (the tutor of Alexander the Great) was a philosopher' and 'the actual tutor of Alexander the Great was a philosopher', if domains are not allowed to vary, is clear. If domains do vary, interesting issues arise if we consider a world w in which Aristotle fails to exist, for decisions have to be made as regards the assignment of truth value to each of those sentences in w: false, or indeterminate. In a world in which Aristotle does not exist, he obviously is not a philosopher, so $< Aristotle, Philosopher* >$ can arguably be said to be false. On the other hand, in that very world Aristotle is not a member of the antiextension of the predicate 'philosopher' and the sentence can be considered to be indeterminate on the grounds that Aristotle is not in the range of applicability of 'philosopher'. And the same goes for $<<$ '*the*', *Actual tutor of Alexander the Great*$* >$, $Philosopher* >$, for there is no individual in w that can be selected by the attributive complex. No matter which way we decide to go, it is natural to think that the two sentences should suffer the same fate. True, it may be argued that the reasoning behind the assignment of falsity or indeterminacy is different for the two sentences. Sure enough, and very interesting, but let us recall that possible worlds semantics is blind to the reasoning and the narratives behind the results: differences in semantic mode of operation are intangible if they do not show up as differences in truth conditions.

However, if we think now in Russellian terms, the sentence ‘the actual tutor of Alexander the Great was a philosopher’ is false, because the claim of existence is false. But as regards ‘dthat (the tutor of Alexander the Great) was a philosopher’, there is still a choice between false and indeterminate. So if we endorse the indeterminacy of the latter, there is a truth conditional difference between ‘dthat’ and ‘actually’.

5 Attributions of existence

Arguably, a truth conditional difference between sentences containing a standard rigid designator and sentences containing an obstinate rigid designator shows up in attributions of existence. Recall that an obstinately rigid designator refers even in worlds in which its designatum fails to exist, whereas a standard rigid designator (or a persistent designator, in Salmon’s terminology) does not designate anything in worlds in which its designatum fails to exist.[8] ‘Dthat’, arguably, operates as an obstinate rigidifier, since the designatum of $dthat(the\,P)$ is the object relevant for the evaluation of sentences in which $dthat(the\,P)$ figures, independently of whether that designatum exists or does not exist at a given index. *The actual* P, on the other hand, is a standard rigid designator. As Scott Soames has noted:

> A dthat-rigidified description, $dthat[the\,x : Fx]$, which designates an object o in the world-state of the context [the actual world], designates o in all world-states, even those in which o does not exist. By contrast, [...] $the\,x : actually\,Fx$ will fail to designate anything at a world-state in which o does not exist. (Soames 2005, 28-29)

Focusing now on attributions of existence, if r is a standard rigid designator, there is a choice as to how sentences of the form r *exists* are evaluated in worlds in which r does not denote. The sentences may be considered false (which appears to be the more natural choice) or they can be interpreted as indeterminate if we abide by the general principle that sentences with denotationless terms are indeterminate. But if r is obstinately rigid, there is no

[8]See (Salmon 1991, 34) for the definition of obstinacy and persistence.

choice; r designates even in worlds in which its designatum fails to exist, and there is no question that in such worlds r *exists* is false.[9]

The difference observed between obstinate rigid designators and standard rigid designators as regards attributions of existence should be expected to occur in the presence of 'dthat' and of 'actually: 'dthat (the tutor of Alexander the Great)' and 'the actual tutor of Alexander the Great' both designate Aristotle in the actual world. But in a world w in which Aristotle does not exist, 'dthat (the tutor of Alexander the Great) exists' would be assigned the truth value False, the same truth value that corresponds to 'Aristotle exists', for those sentences express a complete and evaluable singular proposition $< Aristotle, Exists* >$. But in such a world, the actualized description fails to denote, and the sentence 'the actual tutor of Alexander the Great exists' can be evaluated as indeterminate or as false. So, a truth conditional difference between sentences of the form *the actual P is Q* and sentences of the form $dthat(the\, P)$ *is* Q may manifest itself.

Observe, however, that the case for a truth-conditional difference depends on accepting a specific policy on the treatment of sentences with denotationless terms. And it depends also on accepting 'exists' as a predicate in the language, a controversial choice that some may find objectionable. If we reject the predicative status of 'exists' and paraphrase its ocurrences in favor of quantification, the sentence 'the actual tutor of Alexander the Great exists' will be as false in w as 'dthat (the tutor of Alexander the Great) exists', since any representation of the logical form of 'the actual tutor of Alexander the Great exists' will start with an existential quantifier ranging over the domain of w, and no individual in w is actually tutor of Alexander. So, even in the case of attributions of existence, it is doubtful that the two operators generate sentences that differ in truth value across possible worlds. In fact, this is not a phenomenon that arises because of the use of 'dthat' and 'actually'. Compare 'Aristotle exists' with 'the offspring of gametes X and Y exists.' Let us say that the description 'the offspring of gametes X and Y' refers rigidly to Aristotle. Whereas the name is obstinately rigid, the description is only standardly rigid, however, the formulas $\exists x x = Aristotle$ and $\exists x[Ox \wedge \forall y(Oy \rightarrow x = y)]$ are both false in a world in which Aristotle does not exist. So the phenomenon just observed generalizes to all obstinate and all standard rigid designators.

Attributions of existence do not conclusively provide a way of discriminat-

[9]See also (Gómez-Torrente 2006, 250-251), and (Besson 2009, sect. 2.4), for related discussions on attributions of existence.

ing between obstinately and standardly rigid designators and, in particular, between 'dthat' and 'actually'. There is, however, a truth conditional difference generated by the two operators, one that has not been so widely discussed, and that it is easy to capture without relying on dubious commitments.

6 A truth-conditional difference

Let us consider the following case: in the actual world @ there are many people that smoke. A description such as 'the smoker' is improper and fails to denote. Suppose now that w is a world in which only one of the actual smokers, let us call him a, exists. a may be or may not be a smoker in w. But in w 'the actual smoker' designates a, for a is the unique individual in the domain of w that satisfies the open formula $actual\ smoker(x)$, in virtue of satisfying $smoker(x)$ in @. Hence 'the actual smoker' fails to denote in @, but it does acquire a denotation in w, for in w there is after all a unique individual in the domain that is selected by the attributive complex 'actual smoker': in w, a is an actual smoker.[10] Resorting to the picture of propositions used by Kaplan, we can say that the proposition expressed by 'the actual smoker is a man', the general proposition $<< \text{`}the\text{'}, Actual\, Smoker* > \ , Man* >$, will be false or indeterminate in @, but will be true in w.

'Dthat (the smoker)', on the other hand, does not designate anybody in @ either, for the description it employs fails, like an ambiguous pointing, to demonstrate a unique individual. But unlike 'the actual smoker', 'dthat (the smoker)' will not acquire a denotation in w. The role of the operator 'dthat' applied to a definite description is to provide the individual designated by the definite description as the element that figures in the computation of truth value at all indices of evaluation. In the case of 'dthat (the smoker)' we are missing that element, so the computation of truth value at w does not even get off the ground.

[10](Soames 2005, 29), crediting Ali Kazmi, mentions the peculiar behaviour of expressions of the form *the actual P* when they are improper in the actual world, since they may acquire different denotations in different worlds. Soames' concerns have to do with how this peculiar behaviour affects the status of 'actually' as a rigidifier. Here we are focusing rather on how the truth-conditional behaviour of 'actually' can be shown to be different from the truth-conditional behaviour of 'dthat'. It is worth noticing that it is in order to avoid the effects of this peculiar behavior that Soames proposes (Soames 2005, fn. 22) an interpretation of *the actual P is Q* that imposes the satisfaction of the condition of uniqueness in the actual world. It is attractive to do so, since in the conditions envisaged 'the actual smoker' would not designate in w because it would not designate in @. But, as we have argued (see section 3, and in particular fn. 6), we think that the interpretation Soames suggests is not correct.

Resorting again to the apparatus of structured propositions we can say that since 'dthat (the smoker)' fails to designate in the actual world, there is no propositional constituent provided. The proposition expressed by 'dthat (the smoker) is a man' should be a proposition of the form $< i, Man* >$, a singular proposition with an object in the subject position, an object whose role is to intervene in the computation of truth value in @ and in other indices. But there is no such object. We might say that 'dthat (the smoker) is a man' does not express a proposition. Or, following a lead by Kaplan,[11] we may say that it expresses a gappy proposition, one that can be evaluated as false or as indeterminate, in @ and also in w. But surely, the proposition in question will not be true in w.

Hence, this case suggests that the two operators can have different truth conditional impact. A more precise representation of the semantics of the two sentences discussed will further clarify the details. Following the strategy of section 3, we will first focus on a language that contains a descriptor and the operators 'dthat' and 'actually'. We will then focus on a language in which the descriptor and the operator 'dthat' are eliminated in favor of ordinary and Hazen's actuality quantifiers.

Let us consider a standard first-order language with identity, a descriptor operator 'ι', the modal operator of actuality 'A' and the operator 'dthat'. Given a variable x, a term t and a formula ϕ, $\iota x\phi$ is a term, $A\phi$ is a formula and $dthat(t)$ is a term. As usual, we will consider the language interpreted on a structure consisting of a set of possible worlds W containing the actual world @, a domain of individuals $D(w)$ for each world $w \in W$ and interpretations for constants and predicates.[12] Clauses for interpretation are the usual ones, with the new operators interpreted as:

$Int(\iota x\phi)^w = i$ iff i is the unique individual in $D(w)$ such that i satisfies ϕ in w; otherwise, $Int(\iota x\phi)^w$ is undefined.

$Int(A\phi)^w = Int(\phi)^@$

$Int(dthat(t))^w = Int(t)^@$

In this language, 'the actual smoker is a man' will be formalized as $M(\iota xASx)$ and 'dthat (the smoker) is a man' as $M(dthat(\iota xSx))$. To see the difference in truth conditions, let us consider the following structure $< W, D, Int >$:

$W = \{@, w\}$

[11] See (Kaplan 1989, 496, fn. 23).

[12] Nothing depends on the specific modal logic, so we will not represent the relation of accessibility between worlds. As above, we will also disregard the assignment function.

$D(@) = \{0, 1\}$
$D(w) = \{0, 2\}$
$Int(S)^@ = \{0, 1\}$ $Int(S)^w = \{2\}$
$Int(M)^@ = \{1\}$ $Int(M)^w = \{0\}$

$\iota xASx$ does not denote in @, because in $D(@)$ there are two smokers, and $\iota xASx$ denotes 0 in w, because 0 is the unique actual smoker in $D(w)$. Therefore, the sentence $M(\iota xASx)$ is either false or undetermined (we leave that unspecified) in @, but it is true in w.

By the semantic clause for 'dthat':

$Int(dthat(\iota xSx))^w = Int(\iota xSx)^@ = i$ if i is the unique individual in $D(@)$ that satisfies Sx in @, and it is undefined otherwise.

It is crucial that in this semantic analysis the relativity to the domain of the world of evaluation disappears, so $dthat(\iota xSx)$ denotes neither in @ nor in w, and the sentence $M(dthat(\iota xSx))$ is either false or undetermined in both worlds. We see then that there is a difference in truth value status in the world w.

Let us now consider a language in which 'dthat' and the descriptor have been eliminated. In this case the sentences 'the actual smoker is a man' and 'dthat (the smoker) is a man' will be translated using our previous formalizations (a') and (a*):

$\exists x[ASx \wedge \forall y(ASy \rightarrow x = y) \wedge Mx]$
$\exists^@ x[ASx \wedge \forall^@ y(ASy \rightarrow x = y) \wedge Mx]$

And following the argument of section 3 it is obvious that the first sentence is true and the second is false at the index of evaluation w.

In order to highlight the special status of this case, let us now consider a different structure with the same constituents as before, with the exception of the interpretation of the predicate S, which is replaced by the following one:

$Int(S)^@ = \{1\}$ $\qquad Int(S)^w = \{2\}$

Now $dthat(\iota xSx)$ denotes 1 in @ and the sentence $M(dthat(\iota xSx))$ is true in @. In w, $dthat(\iota xSx)$ still denotes 1, but since 1 is not a member of the domain $D(w)$, the truth value of $M(dthat(\iota xSx))$ depends on decisions about the truth value of sentences in which a term denotes an object that fails to exist in the world in question. As it was the case in the previous structure, the description $\iota xASx$ designates 1 in @, but now it fails to designate in w, making the sentence $M(\iota xASx)$ true in @ and either false or undetermined in w, depending on general policies about the truth-value status of sentences with denotationless terms.[13]

If our policy on how to treat sentences with denotationless terms differs from our policy as regards how to treat sentences in which the denotation of a term fails to exist, then in w we would assign false to one of the sentences and undetermined to the other. A truth conditional difference is revealed in this case, but that difference only shows up because of decisions about semantic phenomena unrelated to the semantic behavior of the operators in question. By contrast, in the first structure, the truth conditional difference is independent of those policies. That, we submit, unmistakably establishes that possible worlds semantics can distinguish between the two operators.[14]

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[13] Note that this structure is a simplified model of sentences such as 'dthat (the tutor of Alexander the Great) was a philosopher' and 'the actual tutor of Alexander the Great was a philosopher' that were discussed in section 4, with 1 playing the role of Aristotle.

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RESEARCH ARTICLE

Semantic Tableau Versions of Some Normal Modal Systems with Propositional Quantifiers

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Abstract: In *Symbolic Logic* (1932), C. I. Lewis developed five modal systems $S1 - S5$. $S4$ and $S5$ are so-called normal modal systems. Since Lewis and Langford's pioneering work many other systems of this kind have been investigated, among them the 32 systems that can be generated by the five axioms T, D, B, 4 and 5. Lewis also discusses how his systems can be augmented by propositional quantifiers and how these augmented logics allow us to express some interesting ideas that cannot be expressed in the corresponding quantifier-free logics. In this paper, I will develop 64 normal modal semantic tableau systems that can be extended by propositional quantifiers yielding 64 extended systems. All in all, we will investigate 128 different systems. I will show how these systems can be used to prove some interesting theorems and I will discuss Lewis's so-called existence postulate and some of its consequences. Finally, I will prove that all normal modal systems are sound and complete and that all systems (including the extended systems) are sound with respect to their semantics. It is left as an open question whether or not the extended systems are complete.

Keywords: C. I. Lewis, Modal logic, Propositional quantifiers, Semantic tableaux.

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1 Introduction

Modal logic deals with modal concepts, such as *necessity*, *possibility* and *contingency*, and with the logical relationships between propositions that include such concepts. Modal logicians study various modal principles, arguments and systems (Blackburn et. al. 2001; 2007, Chellas 1980, Fitting and Mendelsohn 1998, Garson 2006, Hughes and Cresswell 1968; 1996, Kracht 1999, Priest 2008). Lewis and Langford's *Symbolic Logic* (1932) marks the beginning of modern, symbolic, modal logic.

The purpose of this paper is to develop 64 so-called normal modal semantic tableau systems (half of them correspond to the 32 axiomatic systems that can be generated by the well-known axioms T, D, B, 4 and 5) and to show how these systems can be augmented by propositional quantifiers. Therefore, we will consider 128 different systems in this paper. The tableau rules $T-T$, $T-D$, $T-B$, $T-4$ and $T-5$ and the 32 systems that can be generated from these rules are well-known. The normal modal tableau system that includes $T-T$ and $T-4$ is deductively equivalent with, that is, includes the same theorems as, Lewis's system $S4$, and the normal modal tableau system that includes $T-T$, $T-B$ and $T-4$ is deductively equivalent with Lewis's system $S5$.[1] All systems in this paper are stronger than Lewis's systems $S1-S3$. The tableau rule $T-F$, which is especially interesting for our purposes, is much less well-known. Hence, all systems that contain this rule deserve extra attention. The propositional part of all systems in this paper is fairly standard, but as far as I know there are no modal tableau systems in the literature that include propositional quantifiers of the kind that is used in our formal language.[2] Hence, all 64 extended systems are new. Furthermore, I will show how these systems can be used to prove some interesting theorems that contain propositional quantifiers and I will discuss Lewis's so-called existence postulate and some of its consequences. According to this postulate, there is some pair of propositions X and Y, so related that X implies nothing about the truth or falsity of Y (Lewis and Langford 1932, 179). This postulate can be symbolised in the following way: $\exists X \exists Y(\neg\Box(X \to Y) \wedge \neg\Box(X \to \neg Y))$. Finally, I will prove that all normal modal systems are sound and complete

[1] In a strict sense this proposition is not true, since I will not use the exact same language as Lewis. However, if we were to use the same language, they would be deductively equivalent. I will ignore such 'trivial' differences in this paper.

[2] However, (Bull 1969) and (Kripke 1959) use tableaux that are vaguely similar to the tableaux I use in this paper.

and that all systems (including the extended systems) are sound with respect to their semantics. It is left as an open question whether or not the extended systems are complete.

Since all extended systems in this paper are new, there are good logical reasons to be interested in our results. There are also several philosophically interesting reasons. In systems with propositional quantifiers we can express many ideas that cannot be expressed in any quantifier-free normal modal systems. We can, for example, symbolise Lewis's existence postulate, from which it follows that there is something that is contingent, that material implication does not coincide with necessary implication, and that there are at least four distinct propositions, among other things. In ordinary normal modal systems, we cannot prove any of these propositions; we cannot even find plausible formalisations of them. Furthermore, the tableau systems are often more user-friendly than their axiomatic counterparts, it is often easier to prove something in a tableau system than in an axiomatic system and it is often easier to derive a sentence from a set of premises. Consequently, there are both good philosophical and technical reasons to be interested in the systems in this paper. (For more information on propositional quantifiers, see, for example (Lewis and Langford 1932, 178−198), (Kripke 1959), (Bull 1969), (Fine 1970), (Kaplan 1970), (Gabbay 1971) and (Gallin 1975).)

2 Syntax

First, we introduce a quantifier-free language. Then we extend this language with propositional quantifiers.

Alphabet. A set of propositional variables P, Q, R, S, T, X, Y, Z, W, P_1, Q_1, R_1, S_1, T_1, X_1, Y_1, Z_1, W_1, P_2, Q_2, R_2, S_2, T_2, X_2, Y_2, Z_2, W_2, $\ldots$; $\perp$ (Falsum) and $\top$ (Verum); primitive truth-functional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication), and $\leftrightarrow$ (material equivalence); modal operators: $\Box$ (necessity) and $\Diamond$ (possibility), and (iv) the brackets), (.

The language $\mathcal{L}$. The language $\mathcal{L}$ is the set of formulas generated by the usual clauses for atomic sentences, Verum and Falsum and propositionally compound sentences, and the following clause: if A is a formula, then $\Box A$

(it is necessary that A) and $\Diamond A$ (it is possible that A) are formulas; nothing else is a formula. $A, B, C, \ldots$ stand for arbitrary formulas. Formulas are also called 'sentences'.

Definitions. $\blacklozenge A$ (it is impossible that A) $=_{df} \neg \Diamond A$ (or $\Box \neg A$); $\blacksquare A$ (it is non-necessary that A) $=_{df} \neg \Box A$; $\triangledown A$ (it is contingent that A) $=_{df} (\Diamond A \land \Diamond \neg A)$; $\blacktriangledown A$ (it is non-contingent that A) $=_{df} \neg \triangledown A$ (or $(\Box A \lor \Box \neg A)$); $(A \circ B)$ (A is consistent with B) $=_{df} \Diamond(A \land B)$; $(A \bullet B)$ (A is inconsistent with B) $=_{df} \neg(A \circ B)$ $(\neg \Diamond(A \land B)$, $\blacklozenge(A \land B)$, or $\Box \neg(A \land B))$, $(A \Rightarrow B)$ (A strictly implies B) $=_{df} \Box(A \to B)$, $(A \Leftrightarrow B)$ (A is strictly equivalent with B) $=_{df} ((\mathrm{A} \Rightarrow B) \land (B \Rightarrow A))$ $(((\Box(A \to B) \land \Box(B \to A))$, or $\Box(A \leftrightarrow B))$.

The language $\mathcal{L}_{Ext}$. We extend $\mathcal{L}$ by adding two propositional quantifiers $\forall$ (everything/for all) and $\exists$ (something/for some) in the usual way. So, if A is any formula and X is any propositional variable, then $\forall X A$ and $\exists X A$ are formulas. We will call this extended language $\mathcal{L}_{Ext}$. Parentheses are usually dropped when it does not lead to any ambiguity.

The concept of a free variable X in a formula A is defined in the usual way. A variable X is free in A if and only if (iff) it has a free occurrence in A. Intuitively, an occurrence of a variable is free in a formula just in case it is not bound by any quantifier. If A does not contain $\forall$ or $\exists$, then every occurrence of X in A is free. An occurrence of X is free in $\neg B$ ($\Box B, \Diamond B$) iff the corresponding occurrence of X is free in B, and an occurrence of X in $A \land B$ is free iff the corresponding occurrence of X in A or B is free, etc. Finally, an occurrence of X is free in $\forall Y B$ ($\exists Y B$) iff the corresponding occurrence of X is free in B and X is distinct from Y. Any variable occurrences in a formula that are not free are said to be bound. Every free occurrence of X in B is bound by $\forall$ ($\exists$) in $\forall X B$ ($\exists X B$).

Let $(A)[B_1/X_1, \ldots, B_n/X_n]$ be the formula that results by simultaneously replacing all free occurrences of the variable X_1 in A by B_1, …, and all free occurrences of the variable X_n in A by B_n. If there are no free occurrences of $X_1, \ldots, X_n$ in A, then $(A)[B_1/X_1, \ldots, B_n/X_n] = A$. We say that B is substitutable for X in A just in case no variable in B gets bound by a quantifier when B is substituted for X in A. In other words, B is substitutable for X in A just in case, for every variable Y, if an occurrence of Y is free in B, then the corresponding occurrence of Y is free when X is replaced by B in A.

Lewis uses the identity sign for necessary equivalence since he thinks that propositions that are necessarily equivalent are identical. We shall also say that two 'propositions' are 'identical' iff they are necessarily equivalent and that they are 'distinct' iff they are not necessarily equivalent. So, $\exists X \exists Y \neg (X \Leftrightarrow Y)$ says that there are two distinct propositions.

3 Semantics

Model. A model $\mathcal{M}$ is a structure $\langle \mathfrak{W}, \mathfrak{R}, \mathfrak{v} \rangle$, where $\mathfrak{W}$ is a (non-empty) set of possible worlds, $\mathfrak{R}$ is a binary accessibility relation between possible worlds, and $\mathfrak{v}$ is a valuation function. Intuitively, '$\mathfrak{R}\omega\omega'$' means that the possible world ω' is accessible from the possible world ω, or that ω 'can see' ω'. Let X be a propositional variable. Then $\mathfrak{v}(X)$ is a subset of $\mathfrak{W}$. Intuitively, $\mathfrak{v}(X)$ is the proposition that X expresses or the set of possible worlds in which X is true.

Truth conditions. The truth conditions for sentences in $\mathcal{L}$ are defined in the usual way. '$\mathcal{M}, \omega \Vdash A$' says that A is true in the possible world ω in the model $\mathcal{M}$. If X is a propositional variable, then $\mathcal{M}, \omega \Vdash X$ iff ω is an element in $\mathfrak{v}(X)$. Verum is true in every possible world in every model and Falsum is false in every possible world in every model. $\mathcal{M}, \omega \Vdash \neg A$ iff it is not the case that $\mathcal{M}, \omega \Vdash A$. $\mathcal{M}, \omega \Vdash A \wedge B$ iff $\mathcal{M}, \omega \Vdash A$ and $\mathcal{M}, \omega \Vdash B$, etc. $\mathcal{M}, \omega \Vdash \Box A$ iff for every ω' in $\mathfrak{W}$ such that $\mathfrak{R}\omega\omega'$: $\mathcal{M}, \omega' \Vdash A$. $\mathcal{M}, \omega \Vdash \Diamond A$ iff for some ω' in $\mathfrak{W}$: $\mathfrak{R}\omega\omega'$ and $\mathcal{M}, \omega' \Vdash A$.

Intuitively, $\forall X A$ is true in a possible world ω iff $A[B/X]$ is true in ω for every sentence B in $\mathcal{L}$, and $\exists X A$ is true in ω iff $A[B/X]$ is true in ω for some sentence B in $\mathcal{L}$. So, the quantifiers are 'substitutional' rather than 'objectual'. For example, they do not vary directly over (sets of) possible worlds (the range is not a (the) set of (all) subsets of (the set of all) possible worlds). To avoid circularity, we only use formulas from $\mathcal{L}$ in our substitutions. To see the potential problem, let $A = \forall X X$ and assume that our substitutions can include any formula whatsoever. Then $A[A/X] = A$, for $\forall X X[\forall X X/X] = \forall X X$. More precisely, the truth conditions for the new sentences in $\mathcal{L}_{Ext}$ are defined in the following way: $\mathcal{M}, \omega \Vdash \forall X A$ iff for every sentence B (that is substitutable for X in A) in $\mathcal{L}$, $\mathcal{M}, \omega \Vdash A[B/X]$. $\mathcal{M}, \omega \Vdash \exists X A$ iff there is some sentence B

(that is substitutable for X in A) in $\mathcal{L}$ such that $\mathcal{M}, \omega \Vdash A[B/X]$.[3]

Validity. A sentence A is valid in a model, $\mathcal{M} \Vdash A$, iff A is true in every possible world ω in $\mathcal{M}$. Let $\mathfrak{M}$ be a class of models. Then A is valid in $\mathfrak{M}$, $\mathfrak{M} \Vdash A$, iff A is valid in every model $\mathcal{M}$ in $\mathfrak{M}$, that is, iff A is true in every possible world ω in every model $\mathcal{M}$ in $\mathfrak{M}$.

Logical consequence. Let A be a sentence, let Γ be a finite set of sentences and let $\mathfrak{M}$ be a class of models. Then, A is a logical consequence of Γ in $\mathfrak{M}$, $\mathfrak{M}, \Gamma \Vdash A$, iff for every model $\mathcal{M}$ in $\mathfrak{M}$ and world ω in $\mathcal{M}$, if all elements of Γ are true in ω in $\mathcal{M}$, then A is true in ω in $\mathcal{M}$. If $\mathfrak{M}, \Gamma \Vdash A$, we also say that Γ entails A in $\mathfrak{M}$ and that the argument from Γ to A is valid in $\mathfrak{M}$. An argument is invalid in $\mathfrak{M}$ iff it is not valid in $\mathfrak{M}$.

3.1 Conditions on models

In this section, I will consider some conditions that might be imposed on the accessibility relation in a model. In Figure 1, I have listed some of the most well-known conditions in the literature on modal logic along with some formulas. If a model $\mathcal{M}$ satisfies $C - T$, then T is valid in $\mathcal{M}$; if a model $\mathcal{M}$ satisfies $C - D$, then D is valid in $\mathcal{M}$, etc. $C - T$, $C - D$, $C - B$, $C - 4$ and $C - 5$ are mentioned in most introductions to modal logic and they require no further discussion. Condition $C - F$ is less well-known. It is especially interesting for our purposes in this paper. We will, for example, see that this condition is incompatible with Lewis's existence postulate.

The conditions mentioned in Figure 1 can be used to obtain a categorisation of the set of all models into various kinds. We shall say that $\mathfrak{M}(C_1, \dots, C_n)$ is the class of all models that satisfy the conditions $C_1, \dots, C_n$. For example, $\mathfrak{M}(C - T, C - B, C - 4)$ is the class of all models that satisfy the conditions $C - T$, $C - B$ and $C - 4$.

[3] Some of the systems in this paper are vaguely similar to some systems mentioned by (Bull 1969) and (Kripke 1959). There are also some similarities between propositional quantification and quantification over individuals. In Section 4.2, we will see that some Barcan-like formulas can be proved in all systems in this paper. This means that the propositional quantifiers act more like so-called possibilist quantifiers than like so-called actualist quantifiers. For more on how to combine modal logic with predicate logic where the quantifiers vary over objects, see, for example, (Barcan 1946), (Carnap 1946), (Corsi 2002), (Fitting and Mendelsohn 1998), (Garson 1984; 2006), (Hintikka 1961), (Hughes and Cresswell 1968; 1996), (Parks 1976), (Priest 2008), (Stalnaker and Thomason 1968), (Thomason 1970) and (Thomason and Stalnaker 1968).

	Conditions on $\mathfrak{R}$		Corresponding formulas
$C-K$	$-$	K	$\Box(P \to Q) \to (\Box P \to \Box Q)$
$C-T$	$\forall x \mathfrak{R}xx$	T	$\Box P \to P$
$C-D$	$\forall x \exists y \mathfrak{R}xy$	D	$\Box P \to \Diamond P$
$C-B$	$\forall x \forall y(\mathfrak{R}xy \to \mathfrak{R}yx)$	B	$P \to \Box \Diamond P$
$C-F$	$\forall x \forall y \forall z((\mathfrak{R}xy \wedge \mathfrak{R}xz) \to y = z)$	F	$\Diamond P \to \Box P$
$C-4$	$\forall x \forall y \forall z((\mathfrak{R}xy \wedge \mathfrak{R}yz) \to \mathfrak{R}xz)$	4	$\Box P \to \Box \Box P$
$C-5$	$\forall x \forall y \forall z((\mathfrak{R}xy \wedge \mathfrak{R}xz) \to \mathfrak{R}yz)$	5	$\Diamond P \to \Box \Diamond P$

Figure 1: Conditions on models

By using this classification of model classes we can define a large set of systems. The set of all sentences in a language that are valid in a class of models $\mathfrak{M}$ is the (logical) system of $\mathfrak{M}$, $S(\mathfrak{M})$. For example, $S(\mathfrak{M}(C-T, C-B, C-4))$ (the system of $\mathfrak{M}(C-T, C-B, C-4)$) is the class of sentences (in our language) that are valid in the class of models that satisfy the conditions $C-T$, $C-B$ and $C-4$.

There are many interesting relationships between the conditions in Figure 1. Most of them are well-known and require no further comments (see Chellas 1980, ch. 4−5 for more on the relationships between some normal modal systems). However, there are some interesting connections between $C-F$ and some other conditions that are worth mentioning.

$C-D$ says that every possible world can see at least one possible world, and $C-F$ says that every possible world can see at most one possible world. Hence, $C-D$ $\forall x \exists y \mathfrak{R}xy$ and $C-F$ $\forall x \forall y \forall z((\mathfrak{R}xy \wedge \mathfrak{R}xz) \to y = z)$ entail the following condition: $\forall x \exists y(\mathfrak{R}xy \wedge \forall z(\mathfrak{R}xz \to y = z))$, which says that every possible world can see exactly one possible world. Hence, every world can see one and exactly one possible world if we assume $C-D$ and $C-F$. In other words, $C-D$ and $C-F$ entail that $\mathfrak{R}$ is a function (that takes one possible world as input and gives one possible world as output). If a model satisfies $C-F$, then $\Diamond P \to \Box P$ and $\Box P \vee \Box \neg P$ are valid in this model, and if a model satisfies $C-D$, then $\Box P \to \Diamond P$ is valid in this model. Hence, if a model satisfies both $C-D$ and $C-F$, $\Diamond P \leftrightarrow \Box P$, $\Diamond \neg P \leftrightarrow \neg \Diamond P$ and $\blacksquare P \leftrightarrow \blacklozenge P$ are valid in this model. Then, the distinctions between possibility and necessity and between non-necessity and impossibility collapse.

$C-T$ says that $\mathfrak{R}$ is reflexive, that is, that every possible world can see itself, and $C-F$ says that every possible world can see at most one possible world. Hence, $C-T$ $\forall x\mathfrak{R}xx$ and $C-F$ $\forall x\forall y\forall z((\mathfrak{R}xy \wedge \mathfrak{R}xz) \rightarrow y=z)$ entail the following condition $\forall x(\mathfrak{R}xx \wedge \forall y(\mathfrak{R}xy \rightarrow y=x))$, which says that every possible world can see itself and nothing but itself. Furthermore, if a model satisfies $\forall x(\mathfrak{R}xx \wedge \forall y(\mathfrak{R}xy \rightarrow y=x))$, then it also satisfies the following condition $\forall x\forall y(\mathfrak{R}xy \leftrightarrow x=y)$. In other words, if every world can see itself and nothing but itself, then a possible world y is accessible from a possible world x iff x is identical to y. So, if a model satisfies $C-T$ and $C-F$, then $P \leftrightarrow \Box P$, $\neg P \leftrightarrow \neg \Diamond P$ and $\neg P \leftrightarrow \blacklozenge P$ are valid in this model. Hence, the distinctions between truth and necessary truth and between falsehood and impossibility collapse. But if $C-T$ holds, then $C-D$ also holds. So, it is also true that if a model satisfies $C-T$ and $C-F$, then $\Diamond P \leftrightarrow \Box P$, $\Diamond \neg P \leftrightarrow \neg \Diamond P$ and $\blacksquare P \leftrightarrow \blacklozenge P$ are valid in this model. It follows that the distinctions between what is true, possible and necessary collapse. That is, the following equivalences hold $P \leftrightarrow \Diamond P \leftrightarrow \Box P$. Likewise, we have $\neg P \leftrightarrow \Diamond \neg P \leftrightarrow \neg \Diamond P \leftrightarrow \blacksquare P \leftrightarrow \blacklozenge P$. The distinctions between what is false, possibly false, impossible and unnecessary also collapse.

4 Proof theory

In this section, I will develop a set of tableau systems. The propositional part of these systems is similar to systems introduced by (Smullyan 1968) and (Jeffrey 1967), and the modal part is similar to systems discussed by (Priest 2008). The new interesting thing is the rules for the propositional quantifiers. For more information about the tableau method and various kinds of tableau systems, see, for example, (D'Agostino, Gabbay, Hähnle and Posegga 1999) and (Fitting and Mendelsohn 1998).

4.1 Tableau rules

4.1.1 Propositional rules

$\neg\neg$	$\wedge$	$\neg\wedge$
$\neg\neg A, i$ $\downarrow$ A, i	$A \wedge B, i$ $\downarrow$ A, i B, i	$\neg(A \wedge B), i$ $\swarrow \searrow$ $\neg A, i \quad \neg B, i$
$\vee$	$\neg\vee$	$\rightarrow$
$A \vee B, i$ $\swarrow \searrow$ $A, i \quad B, i$	$\neg(A \vee B), i$ $\downarrow$ $\neg A, i$ $\neg B, i$	$A \rightarrow B, i$ $\swarrow \searrow$ $\neg A, i \quad B, i$
$\neg \rightarrow$	$\leftrightarrow$	$\neg \leftrightarrow$
$\neg(A \rightarrow B), i$ $\downarrow$ A, i $\neg B, i$	$A \leftrightarrow B, i$ $\swarrow \searrow$ $A, i \quad \neg A, i$ $B, i \quad \neg B, i$	$\neg(A \leftrightarrow B), i$ $\swarrow \searrow$ $A, i \quad \neg A, i$ $\neg B, i \quad B, i$

Figure 2: Propositional rules

4.1.2 Basic modal rules

$\Box$	$\Diamond$	$\neg\Box$	$\neg\Diamond$
$\Box A, i$ irj $\downarrow$ A, j	$\Diamond A, i$ $\downarrow$ irj A, j	$\neg\Box A, i$ $\downarrow$ $\Diamond\neg A, i$	$\neg\Diamond A, i$ $\downarrow$ $\Box\neg A, i$

Figure 3: Basic modal rules

4.1.3 Quantifier rules

$\forall$	$\exists$	$\neg\forall$	$\neg\exists$
$\forall XA, i$	$\exists XA, i$	$\neg\forall XA, i$	$\neg\exists XA, i$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$A[B/X], i$	$A[Y/X], i$	$\exists X\neg A, i$	$\forall X\neg A, i$

Figure 4: Propositional quantifier rules

Note that in ($\forall$), B is any (quantifier-free) formula (in $\mathcal{L}$) that is substitutable for X in A; and in ($\exists$), Y is a propositional variable that is new to the branch.

4.1.4 Accessibility rules

$T-D$	$T-T$	$T-B$	$T-F$	$T-4$	$T-5$
i	i	irj	irj	irj	irj
$\downarrow$	$\downarrow$	$\downarrow$	irk	jrk	irk
irj	iri	jri	$\downarrow$	$\downarrow$	$\downarrow$
where j is new			$j=k$	irk	jrk

Figure 5: Accessibility rules

4.1.5 The CUT-rule and the identity-rules

CUT	$T-Ii$	$T-Iii$
$*$	$A(i)$	$A(j)$
$\swarrow \quad \searrow$	$i=j$	$i=j$
$A, i \quad \neg A, i$	$\downarrow$	$\downarrow$
for every A and i	$A(j)$	$A(i)$

Figure 6: The CUT-rule and the identity-rules

The identity rules should be interpreted in the following way. $A(i)$ is a line in a tableau that includes 'i', and $A(j)$ is like $A(i)$ except that 'i' is replaced by

'j'. That is, if $A(i)$ is A, i, then $A(j)$ is A, j; if $A(i)$ is kri, then $A(j)$ is krj; if $A(i)$ is $i = k$, then $A(j)$ is $j = k$, etc. '$*$' in CUT indicates that CUT is a rule without any 'premises'.

4.2 Tableau systems and some basic proof-theoretical concepts

A tableau system is a set of tableau rules. A normal modal tableau system is a tableau system that includes all propositional rules and all basic modal rules. The smallest normal modal tableau system without any accessibility rules is called K. By adding various additional accessibility rules, we obtain a large class of stronger systems. By combining the different rules in various ways, we can construct 64 different normal modal tableau systems (many of these are deductively equivalent, that is, include the same theorems). To make some proofs easier, it is often useful to add the CUT-rule. The identity rules are included in every system that contains $T - F$. If S is a normal modal system, then S_{Ext} is the normal modal system S extended by the quantifier rules. There are 64 extended systems of this kind. All in all, we consider 128 different systems in this paper. Many of these are deductively equivalent, that is, contain the same theorems.

Let $T_1, \ldots, T_n$ be the (normal modal) tableau system that includes the tableau rules $T_1, \ldots, T_n$. The initial T may be omitted if it is clear that we are talking about a tableau system. Then, $TB4$ is the (normal modal) tableau system that includes the tableau rules $T - T$, $T - B$ and $T - 4$, etc.

The concepts of semantic tableau, branch, open and closed branch, etc. are essentially defined as usual. A (semantic) tableau is a tree-like structure where the nodes in the tree have the following form: A, i, where A is a formula in our language and i is in $\{0, 1, 2, 3, \ldots\}$, or something of the form irj, or $i = j$ where i, j is in $\{0, 1, 2, 3, \ldots\}$. A branch in a tableau is closed iff there is a formula A and a number i, such that both A, i and $\neg A, i$ occur on the branch or if we have $\neg\top, i$ or $\bot, i$ on the branch; it is open just in case it is not closed. A tableau itself is closed iff every branch in it is closed; it is open iff it is not closed.

Let S be any system in this paper and let an S-tableau be a tableau generated in accordance with the rules in S. Furthermore, let A be a sentence and let Γ be a finite set of sentences. '$\vdash_S A$' says that A is a theorem in S and

'$\Gamma \vdash_S A$' says that A is derivable from Γ in S.

Proof in a system. A proof of A in S is a closed S-tableau that starts with $\neg A, 0$.

Theorem in a system. A sentence A is a theorem in S or provable in S, $\vdash_S A$, iff there is a proof of A in S, that is, iff there is a closed S-tableau that starts with $\neg A, 0$.

Derivation in a system. A derivation in the system S of the sentence A from the set of sentences Γ, is a closed S-tableau whose initial list comprises $B, 0$ for every B in Γ and $\neg A, 0$. The sentences in Γ are the premises of the derivation and A is called the conclusion of the derivation. The initial list of a tableau consists of the first nodes in this tableau whose 'satisfiability' we are testing.

Proof-theoretic consequence in a system. A sentence A is a proof-theoretic consequence of a set of sentences Γ in S or A is derivable from Γ in $S, \Gamma \vdash_S A$, iff there is a derivation of A in S from Γ, that is, iff there is a closed S-tableau whose initial list comprises $B, 0$ for every B in Γ and $\neg A, 0$.

4.3 Some theorems

We are now in a position to prove some propositions. I will go through some theorems that appear in (Lewis and Langford 1932, 178−198) to illustrate how to use our systems and especially the new quantifier rules. I will also consider some additional theorems that are not discussed by Lewis and Langford.

First, let us mention some important relationships between some fundamental modal concepts. All of the following sentences are theorems in every system in this paper (some hold by definition): $\lozenge P \leftrightarrow (P \circ P)$, $\blacklozenge P \leftrightarrow \neg(P \circ P)$, $\Box P \leftrightarrow \neg(\neg P \circ \neg P)$, $\blacksquare P \leftrightarrow (\neg P \circ \neg P)$, $(P \bullet Q) \leftrightarrow \neg(P \circ Q)$, $(P \Rightarrow Q) \leftrightarrow \neg(P \circ \neg Q)$, $(P \Leftrightarrow Q) \leftrightarrow (\neg(P \circ \neg Q) \wedge \neg(Q \circ \neg P))$, $\lozenge P \leftrightarrow \neg(P \bullet P)$, $\blacklozenge P \leftrightarrow (P \bullet P)$, $\Box P \leftrightarrow (\neg P \bullet \neg P)$, $\blacksquare P \leftrightarrow \neg(\neg P \bullet \neg P)$, $(P \circ Q) \leftrightarrow \neg(P \bullet Q)$, $(P \Rightarrow Q) \leftrightarrow (P \bullet \neg Q)$, $(P \Leftrightarrow Q) \leftrightarrow ((P \bullet \neg Q) \wedge (Q \bullet \neg P))$, $\lozenge P \leftrightarrow \neg(P \Rightarrow \neg P)$, $\blacklozenge P \leftrightarrow (P \Rightarrow \neg P)$, $\Box P \leftrightarrow (\neg P \Rightarrow P)$, $\blacksquare P \leftrightarrow \neg(\neg P \Rightarrow P)$, $(P \circ Q) \leftrightarrow \neg(P \Rightarrow \neg Q)$, $(P \bullet Q) \leftrightarrow (P \Rightarrow \neg Q)$, $(P \Leftrightarrow Q) \leftrightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$, $\lozenge P \leftrightarrow \neg \Box \neg P$, $\blacklozenge P \leftrightarrow \Box \neg P$, $\blacksquare P \leftrightarrow \neg \Box P$,

$(P \circ Q) \leftrightarrow \neg\Box\neg(P \wedge Q)$, $(P \bullet Q) \leftrightarrow \Box\neg(P \wedge Q)$, $(P \Rightarrow Q) \leftrightarrow \Box(P \rightarrow Q)$, $(P \Leftrightarrow Q) \leftrightarrow \Box(P \leftrightarrow Q)$, $\blacklozenge P \leftrightarrow \neg\Diamond P$, $\Box P \leftrightarrow \neg\Diamond\neg P$, $\blacksquare P \leftrightarrow \Diamond\neg P$, $(P \circ Q) \leftrightarrow \Diamond(P \wedge Q)$, $(P \bullet Q) \leftrightarrow \neg\Diamond(P \wedge Q)$, $(P \Rightarrow Q) \leftrightarrow \neg\Diamond\neg(P \rightarrow Q)$, $(P \Leftrightarrow Q) \leftrightarrow \neg\Diamond\neg(P \leftrightarrow Q)$, $\Diamond P \leftrightarrow \neg\blacklozenge P$, $\Box P \leftrightarrow \blacklozenge\neg P$, $\blacksquare P \leftrightarrow \neg\blacklozenge\neg P$, $(P \circ Q) \leftrightarrow \neg\blacklozenge(P \wedge Q)$, $(P \bullet Q) \leftrightarrow \blacklozenge(P \wedge Q)$, $(P \Rightarrow Q) \leftrightarrow \blacklozenge\neg(P \rightarrow Q)$, $(P \Leftrightarrow Q) \leftrightarrow \blacklozenge\neg(P \leftrightarrow Q)$.

Lewis observes that we can prove that there exists at least one proposition which is true and that there exists at least one false proposition. We can also show that not everything is true ($\neg\forall XX$) and that not everything is false ($\neg\forall X\neg X$). Let us verify the claim that there is something true and that there is something false.

$\exists XX$. Something is true.
(1) $\neg\exists XX$, 0
(2) $\forall X\neg X$, 0 [1, $\neg\exists$]
(3) $\neg(P \vee \neg P)$, 0 [2, $\forall$ $(\neg X)[P \vee \neg P/X]$]
(4) $\neg P$, 0 [3, $\neg\vee$]
(5) $\neg\neg P$, 0 [3, $\neg\vee$]
(6) $*$ [4, 5]

$\exists X\neg X$. Something is false.
(1) $\neg\exists X\neg X$, 0
(2) $\forall X\neg\neg X$, 0 [1, $\neg\exists$]
(3) $\neg\neg(P \wedge \neg P)$, 0 [2, $\forall$ $(\neg\neg X)[P \wedge \neg P/X]$]
(4) $P \wedge \neg P$, 0 [3, $\neg\neg$]
(5) P, 0 [4, $\wedge$]
(6) $\neg P$, 0 [4, $\wedge$]
(7) $*$ [5, 6]

Here are some other theorems that do not contain any modal operators. All of these sentences can be proved in all our systems. $\forall X(\bot \rightarrow X)$: *Falsum* (a contradiction) materially implies anything. $\forall X(X \rightarrow \top)$: Everything materially implies *Verum* (something that is logically true). $\forall X(X \rightarrow \forall Y(Y \rightarrow X))$: If something is true it is materially implied by anything. $\forall X(\forall Y(Y \rightarrow X) \rightarrow X)$: If something is materially implied by anything, it is true. $\forall X(X \leftrightarrow \forall Y(Y \rightarrow X))$: Something is true iff it is materially implied by anything. $\forall X(\neg X \rightarrow \forall Y(X \rightarrow Y))$: If something is false, it materially implies everything. $\forall X(\forall Y(X \rightarrow Y) \rightarrow \neg X)$: If something materially implies everything, it is false. $\forall X(\neg X \leftrightarrow \forall Y(X \rightarrow Y))$: Something is false iff it materially implies everything.

All of the following sentences are theorems in every system in this paper: $\forall X(\exists Y((Y \to X) \wedge (\neg Y \to X)) \to X)$, $\forall X(X \to \exists Y((Y \to X) \wedge (\neg Y \to X)))$, $\forall X(X \leftrightarrow \exists Y((Y \to X) \wedge (\neg Y \to X)))$ (X is true iff there is a Y such that both Y and the negation of Y implies X), $\forall X(\exists Y((X \to Y) \wedge (X \to \neg Y)) \to \neg X)$, $\forall X(\neg X \to \exists Y((X \to Y) \wedge (X \to \neg Y)))$, $\forall X(\neg X \leftrightarrow \exists Y((X \to Y) \wedge (X \to \neg Y)))$ (X is false iff there is a Y such that X implies both Y and the negation of Y).

Let A be a formula that does not contain any free occurrence of X. Then all equivalences of the following forms are provable: $\forall X(X \to A) \leftrightarrow (\exists XX \to A)$, $\forall X(A \to X) \leftrightarrow (A \to \forall XX)$, $\exists X(A \to X) \leftrightarrow (A \to \exists XX)$. I will show the first and leave the rest to the reader.

$\forall X(X \to A) \leftrightarrow (\exists XX \to A)$
Left to right $\forall X(X \to A) \to (\exists XX \to A)$
(1) $\neg(\forall X(X \to A) \to (\exists XX \to A))$, 0
(2) $\forall X(X \to A)$, 0 [1, $\neg \to$]
(3) $\neg(\exists XX \to A)$, 0 [1, $\neg \to$]
(4) $\exists XX$, 0 [3, $\neg \to$]
(5) $\neg A$, 0 [3, $\neg \to$]
(6) Y, 0 [4, $\exists$ $(X)[Y/X]$]
(7) $Y \to A$, 0 [2, $\forall$ $(X \to A)[Y/X]$]
↙ ↘
(8) $\neg Y$, 0 [7, $\to$] (9) A, 0 [7, $\to$]
(10) $*$ [6, 8] (11) $*$ [5, 9]

Right to left $(\exists XX \to A) \to \forall X(X \to A)$
(1) $\neg((\exists XX \to A) \to \forall X(X \to A))$, 0
(2) $\exists XX \to A$, 0 [1, $\neg \to$]
(3) $\neg\forall X(X \to A)$, 0 [1, $\neg \to$]
(4) $\exists X\neg(X \to A)$, 0 [3, $\neg\forall$]
(5) $\neg(Y \to A)$, 0 [4, $\exists$ $(\neg(X \to A))[Y/X]$]
(6) Y, 0 [5, $\neg \to$]
(7) $\neg A$, 0 [5, $\neg \to$]
↙ ↘
(8) $\neg\exists XX$, 0 [2, $\to$] (9) A, 0 [2, $\to$]
(10) $\forall X\neg X$, 0 [8, $\neg\exists$] (11) $*$ [7, 9]
(12) $\neg Y$, 0 [10, $\forall$ $(\neg X)[Y/X]$]
(13) $*$ [6, 12]

Lewis proves that there are at least two propositions that are distinct (not necessarily equivalent). We cannot prove this claim in every system in this paper. However, we can show that it holds in every system that includes $T-T$

(or $T-D$). $\exists X \exists Y \neg (X \Leftrightarrow Y)$ is by definition equivalent with $\exists X \exists Y \neg \Box (X \leftrightarrow Y)$. So, to prove the former, we show the latter.

$\exists X \exists Y \neg \Box (X \leftrightarrow Y)$. There are at least two propositions that are not necessarily equivalent.

(1) $\neg \exists X \exists Y \neg \Box (X \leftrightarrow Y)$, 0
(2) $\forall X \neg \exists Y \neg \Box (X \leftrightarrow Y)$, 0 [1, $\neg\exists$]
(3) $\neg \exists Y \neg \Box (\top \leftrightarrow Y)$, 0 [2, $\forall$ $(\neg \exists Y \neg \Box (X \leftrightarrow Y))[\top / X]$]
(4) $\forall Y \neg\neg \Box (\top \leftrightarrow Y)$, 0 [3, $\neg\exists$]
(5) $\neg\neg \Box (\top \leftrightarrow \bot)$, 0 [4, $\forall$ $(\neg\neg \Box (\top \leftrightarrow Y))[\bot / Y]$]
(6) $\Box (\top \leftrightarrow \bot)$, 0 [5, $\neg\neg$]
(7) $0r0$ [T]
(8) $\top \leftrightarrow \bot$, 0 [6, 7, $\Box$]

↙	↘
(9) $\top$, 0 [8, $\leftrightarrow$]	(10) $\neg\top$, 0 [8, $\leftrightarrow$]
(11) $\bot$, 0 [8, $\leftrightarrow$]	(12) $\neg\bot$, 0 [8, $\leftrightarrow$]
(13) $*$ [11]	(14) $*$ [10]

This proof depends on T at Step 7. This step is essential (however, in systems that include D it can be replaced by '$0r1$' if (8)−(12) are modified in an obvious way). If a possible world cannot see any world, then everything is necessary in this world. Hence, all equivalences are necessary in this world.

In every system in this paper, we can show that any two necessary propositions are necessarily equivalent and hence not distinct, $\forall X \forall Y ((\Box X \wedge \Box Y) \rightarrow (X \Leftrightarrow Y))$, and that any two impossible propositions are necessarily equivalent and hence not distinct, $\forall X \forall Y ((\blacklozenge X \wedge \blacklozenge Y) \rightarrow (X \Leftrightarrow Y))$. It is left to the reader to verify this claim.

We have seen that there is something true and that there is something false. We can also show that some propositions are necessarily true ($\exists X \Box X$), that some propositions are possibly true ($\exists X \Diamond X$), that some propositions are necessarily false ($\exists X \Box \neg X$), that some propositions cannot be true ($\exists X \neg \Diamond X$), that some propositions are possibly false ($\exists X \Diamond \neg X$), that some propositions are not necessarily true ($\exists X \neg \Box X$), that some propositions are not necessarily false ($\exists X \neg \Box \neg X$), and that some propositions cannot be false ($\exists X \neg \Diamond \neg X$). The proof of $\exists X \Diamond X$ requires T. It can also be proved in systems with D. If a possible world cannot see any possible world, then nothing is possible in this world. Then we cannot conclude that there is something possible in this world. Similar things can be said about $\exists X \Diamond \neg X$, $\exists X \neg \Box X$ and $\exists X \neg \Box \neg X$. All the other propositions hold in every system. I will establish that there are necessary truths and leave the other proofs to the reader.

$\exists X \Box X$. Some propositions are necessarily true. There are necessary truths.

(1) $\neg\exists X \Box X$, 0
(2) $\forall X \neg \Box X$, 0 [1, $\neg\exists$]
(3) $\neg \Box(P \leftrightarrow P)$, 0 [2, $\forall$ $(\neg \Box X)[P \leftrightarrow P/X]$]
(4) $\Diamond \neg(P \leftrightarrow P)$, 0 [3, $\neg \Box$]
(5) $0r1$ [4, $\Diamond$]
(6) $\neg(P \leftrightarrow P)$, 1 [4, $\Diamond$]

↙	↘
(7) P, 1 [6, $\neg \leftrightarrow$]	(8) $\neg P$, 1 [6, $\neg \leftrightarrow$]
(9) $\neg P$, 1 [6, $\neg \leftrightarrow$]	(10) P, 1 [6, $\neg \leftrightarrow$]
(11) $*$ [7, 9]	(12) $*$ [8, 10]

Lewis observes that no proposition is equivalent to its own negation. We can prove this proposition in every system that includes T (or D). $\forall X \neg(X \Leftrightarrow \neg X)$ is by definition equivalent with $\forall X \neg \Box(X \leftrightarrow \neg X)$.

In our systems, we can prove several Barcan-like formulas. All of the following sentences are provable in every system in this paper: if everything is necessarily true, then it is necessary that everything is true ($\forall X \Box X \rightarrow \Box \forall X X$), if it is necessary that everything is true, then everything is necessarily true ($\Box \forall X X \rightarrow \forall X \Box X$), it is necessary that everything is true iff everything is necessarily true ($\Box \forall X X \leftrightarrow \forall X \Box X$), if something is possibly true, then it is possible that something is true ($\exists X \Diamond X \rightarrow \Diamond \exists X X$), if it is possible that something is true, then something is possibly true ($\Diamond \exists X X \rightarrow \exists X \Diamond X$), it is possible that something is true iff something is possibly true ($\Diamond \exists X X \leftrightarrow \exists X \Diamond X$), if something is necessarily true, then it is necessary that something is true ($\exists X \Box X \rightarrow \Box \exists X X$), if it is possible that everything is true, then everything is possibly true ($\Diamond \forall X X \rightarrow \forall X \Diamond X$). I will prove the first and leave the rest to the reader.

$\forall X \Box X \rightarrow \Box \forall X X$. If everything is necessarily true, then it is necessary that everything is true.

(1) $\neg(\forall X \Box X \rightarrow \Box \forall X X)$, 0
(2) $\forall X \Box X$, 0 [1, $\neg \rightarrow$]
(3) $\neg \Box \forall X X$, 0 [1, $\neg \rightarrow$]
(4) $\Diamond \neg \forall X X$, 0 [3, $\neg \Box$]
(5) $0r1$ [4, $\Diamond$]
(6) $\neg \forall X X$, 1 [4, $\Diamond$]
(7) $\exists X \neg X$, 1 [6, $\neg\forall$]
(8) $\neg Y$, 1 [7, $\exists$ $(\neg X)[Y/X]$]
(9) $\Box Y$, 0 [2, $\forall$ $(\Box X)[Y/X]$]
(10) Y, 1 [5, 9, $\Box$]
(11) $*$ [8, 10]

Here are some other theorems that can be proved in our systems: $\exists Y(P \wedge Y) \rightarrow P$, $P \leftrightarrow \exists Y(P \wedge Y)$, $\neg P \rightarrow \forall Y \neg(P \wedge Y)$, $\exists Y((P \rightarrow Y) \wedge (Y \rightarrow R)) \rightarrow (P \rightarrow R)$, $P \rightarrow \forall Y(Y \rightarrow P)$, $P \leftrightarrow \forall Y(Y \rightarrow P)$, $\neg P \rightarrow \forall Y(P \rightarrow Y)$, $\neg P \leftrightarrow \forall Y(P \rightarrow Y)$, $\neg \Diamond P \rightarrow \exists Y(\Box(P \rightarrow Y) \wedge \Box(P \rightarrow \neg Y))$, $\exists Y(\Box(P \rightarrow Y) \wedge \Box(P \rightarrow \neg Y)) \rightarrow \neg \Diamond P$, $\Diamond P \rightarrow \forall Y(\neg \Box(P \rightarrow Y) \vee \neg \Box(P \rightarrow \neg Y))$, $\forall Y(\neg \Box(P \rightarrow Y) \vee \neg \Box(P \rightarrow \neg Y)) \rightarrow \Diamond P$, $\exists Y(\Box(Y \rightarrow P) \wedge \Box(\neg Y \rightarrow P)) \rightarrow \Box P$, $\Box P \rightarrow \exists Y(\Box(Y \rightarrow P) \wedge \Box(\neg Y \rightarrow P))$.

Now, let us consider some theorems that include the consistency operator (see Lewis and Langford, p. 196). $\exists X((X \Rightarrow Q) \wedge (X \circ R)) \rightarrow (Q \circ R)$: If there is an X such that X necessarily implies Q and X is consistent with R, then Q is consistent with R. $\exists Y((P \Rightarrow Y) \wedge \neg(Y \circ R)) \rightarrow \neg(P \circ R)$: If there is a Y such that P necessarily implies Y and Y is inconsistent with R, then P is inconsistent with R. $\exists X \exists Y((X \Rightarrow R) \wedge (Y \Rightarrow S) \wedge (X \circ Y)) \rightarrow (R \circ S)$: If there is an X and a Y such that X necessarily implies R and Y necessarily implies S and X is consistent with Y, then R is consistent with S. $\exists Y((P \Rightarrow Y) \wedge (P \Rightarrow \neg Y)) \rightarrow \neg(P \circ P)$, $\exists X((X \Rightarrow Q) \wedge (X \circ X)) \rightarrow (Q \circ Q)$, $\exists Y((P \Rightarrow Y) \wedge \neg(Y \circ Y)) \rightarrow \neg(P \circ P)$, $(P \circ P) \rightarrow \forall Y \neg((P \Rightarrow Y) \wedge (P \Rightarrow \neg Y))$, $(P \circ P) \rightarrow \forall Y((P \circ Y) \vee (P \circ \neg Y))$, $\exists X \exists Y((P \Rightarrow X) \wedge (Q \Rightarrow Y) \wedge \neg(X \circ Y)) \rightarrow \neg(P \circ Q)$: If there is an X and a Y such that P necessarily implies X and Q necessarily implies Y and X is inconsistent with Y, then P is inconsistent with Q. By definition the last formula is equivalent with the following sentence: $\exists X \exists Y(\Box(P \rightarrow X) \wedge \Box(Q \rightarrow Y) \wedge \neg \Diamond(X \wedge Y)) \rightarrow \neg \Diamond(P \wedge Q)$. So, to prove the former, it is sufficient to establish the latter. Let us prove this formula ('MP' is an abbreviation of 'Modus Ponens', which is a derived rule in our systems).

$\exists X \exists Y(\Box(P \rightarrow X) \wedge \Box(Q \rightarrow Y) \wedge \neg \Diamond(X \wedge Y)) \rightarrow \neg \Diamond(P \wedge Q)$.
(1) $\neg(\exists X \exists Y((\Box(P \rightarrow X) \wedge \Box(Q \rightarrow Y)) \wedge \neg \Diamond(X \wedge Y)) \rightarrow \neg \Diamond(P \wedge Q))$, 0
(2) $\exists X \exists Y((\Box(P \rightarrow X) \wedge \Box(Q \rightarrow Y)) \wedge \neg \Diamond(X \wedge Y))$, 0 [1, $\neg \rightarrow$]
(3) $\neg\neg \Diamond(P \wedge Q)$, 0 [1, $\neg \rightarrow$]
(4) $\Diamond(P \wedge Q)$, 0 [3, $\neg\neg$]
(5) $\exists Y((\Box(P \rightarrow Z) \wedge \Box(Q \rightarrow Y)) \wedge \neg \Diamond(Z \wedge Y))$, 0 [2, $\exists$]
(6) $(\Box(P \rightarrow Z) \wedge \Box(Q \rightarrow W)) \wedge \neg \Diamond(Z \wedge W)$, 0 [5, $\exists$]
(7) $\Box(P \rightarrow Z) \wedge \Box(Q \rightarrow W)$, 0 [6, $\wedge$]
(8) $\neg \Diamond(Z \wedge W)$, 0 [6, $\wedge$]
(9) $\Box(P \rightarrow Z)$, 0 [7, $\wedge$]
(10) $\Box(Q \rightarrow W)$, 0 [7, $\wedge$]
(11) $\Box \neg(Z \wedge W)$, 0 [8, $\neg \Diamond$]
(12) $0r1$ [4, $\Diamond$]
(13) $P \wedge Q$, 1 [4, $\Diamond$]

(14) P, 1 [13, $\wedge$]
(15) Q, 1 [13, $\wedge$]
(16) $P \to Z$, 1 [9, 12, $\Box$]
(17) $Q \to W$, 1 [10, 12, $\Box$]
(18) Z, 1 [14, 16, MP]
(19) W, 1 [15, 17, MP]
(20) $\neg(Z \wedge W)$, 1 [11, 12, $\Box$]
$\swarrow$ $\searrow$
(21) $\neg Z$, 1 [20, $\neg\wedge$] (22) $\neg W$, 1 [20, $\neg\wedge$]
(23) $*$ [18, 21] (24) $*$ [19, 22]

Here are some other interesting theorems that can be proved in our systems.

$\forall X(\Box X \to \forall Y \Box(Y \to X))$: If something is necessarily true it is necessarily implied by anything. $\forall X(\forall Y \Box(Y \to X) \to \Box X)$: If something is necessarily implied by anything, it is necessarily true. $\forall X(\Box X \leftrightarrow \forall Y \Box(Y \to X))$: Something is necessarily true iff it is necessarily implied by anything. $\forall X(\blacklozenge X \to \forall Y \Box(X \to Y))$: If something is impossible, it necessarily implies everything. $\forall X(\forall Y \Box(X \to Y) \to \blacklozenge X)$: If something necessarily implies everything, it is impossible. $\forall X(\blacklozenge X \leftrightarrow \forall Y \Box(X \to Y))$: Something is impossible iff it necessarily implies everything. $\forall X(\Diamond X \to \neg\forall Y \Box(X \to Y))$: If something is possible, it is not the case that it necessarily implies everything. $\forall X(\neg\forall Y \Box(X \to Y) \to \Diamond X)$: If it is not the case that something necessarily implies everything, then it is possible. $\forall X(\Diamond X \leftrightarrow \neg\forall Y \Box(X \to Y))$: X is possible iff it is not the case that X necessarily implies everything. $\forall X(\Box X \leftrightarrow \forall Y \Box(\neg X \to Y))$: It is necessary that X iff the negation of X implies everything. $\forall X(\Diamond X \leftrightarrow \exists Y \neg\Box(X \to Y))$: X is possible iff there is something X does not necessarily imply. $\forall X(\blacklozenge X \leftrightarrow \neg\exists Y \neg\Box(X \to Y))$: X is impossible iff there is nothing X does not necessarily imply. $\forall X(\Box X \leftrightarrow \neg\exists Y \neg\Box(\neg X \to Y))$: X is necessary iff there is nothing that the negation of X does not imply. The theorems in this paragraph tell us something about the necessary and sufficient conditions for something to be necessary, possible and impossible. Our next theorems also consider such conditions.

$\forall X(\exists Y(\Box(Y \to X) \wedge \Box(\neg Y \to X)) \to \Box X)$: If there is a proposition such that both it and its negation necessarily implies X, then X is necessary. $\forall X(\Box X \to \exists Y(\Box(Y \to X) \wedge \Box(\neg Y \to X)))$: If X is necessary, then there is a proposition such that both it and its negation necessarily implies X. $\forall X(\Box X \leftrightarrow \exists Y(\Box(Y \to X) \wedge \Box(\neg Y \to X)))$: Something X is necessary iff there is a proposition such that both it and its negation necessarily implies X. $\forall X(\exists Y(\Box(X \to Y) \land \Box(X \to \neg Y)) \to \blacklozenge X)$: If there is a proposition such that X necessarily implies both it and its negation, then X is impossible. $\forall X(\blacklozenge X \to \exists Y(\Box(X \to Y) \land \Box(X \to \neg Y)))$: If X is impossible, then there is a proposition such that X necessarily implies both it and its negation. $\forall X(\blacklozenge X \leftrightarrow \exists Y(\Box(X \to Y) \land \Box(X \to \neg Y)))$: X is impossible iff there is a proposition such that X necessarily implies both it and its negation. $\forall X(\neg\exists Y(\Box(X \to Y) \land \Box(X \to \neg Y)) \to \Diamond X)$: If there is no proposition such that X necessarily implies both it and its negation, then X is possible. $\forall X(\Diamond X \to \neg\exists Y(\Box(X \to Y) \land \Box(X \to \neg Y)))$: If X is possible, then there is no proposition such that X necessarily implies both it and its negation. $\forall X(\Diamond X \leftrightarrow \neg\exists Y(\Box(X \to Y) \land \Box(X \to \neg Y)))$: X is possible iff there is no proposition such that X necessarily implies both it and its negation. $\forall X(\Box X \leftrightarrow \forall Y(\Box(Y \to X) \land \Box(\neg Y \to X)))$, $\forall X(\Diamond X \leftrightarrow \neg\forall Y(\Box(X \to Y) \land \Box(X \to \neg Y)))$, $\forall X(\Diamond X \leftrightarrow \forall Y(\neg\Box(X \to Y) \lor \neg\Box(X \to \neg Y)))$, $\forall X(\Diamond X \leftrightarrow \exists Y(\neg\Box(X \to Y) \lor \neg\Box(X \to \neg Y)))$, $\forall X(\blacklozenge X \leftrightarrow \forall Y(\Box(X \to Y) \land \Box(X \to \neg Y)))$.

We can prove the following sentences in our systems: $\forall X(\exists Y(\Box(X \to Y) \land \Box(X \to \neg Y)) \leftrightarrow \forall Y(\Box(X \to Y) \land \Box(X \to \neg Y)))$, there is a proposition such that X necessarily implies both it and its negation iff it is true of every proposition that X necessarily implies both it and its negation. $\forall X(\exists Y(\Box(Y \to X) \land \Box(\neg Y \to X)) \leftrightarrow \forall Y(\Box(Y \to X) \land \Box(\neg Y \to X)))$, there is a proposition such that both it and its negation implies X iff it is true of every proposition that both it and its negation implies X.

In systems with T, we can prove the following sentences: $\Box\forall X\Box X \to \Box\forall XX$ and $\Box\forall X\Box X \to \forall X\Box X$. In systems with 4, we can prove the following sentences: $\forall X\Box X \to \Box\forall X\Box X$ and $\Box\forall XX \to \Box\forall X\Box X$. Finally, in systems with T and 4 all of the following sentences are equivalent: $\Box\forall XX$, $\forall X\Box X$ and $\Box\forall X\Box X$.

Now, let us turn to Lewis's existence postulate.

4.4 Lewis and Langford's existence postulate

According to Lewis's so-called existence postulate, which categorically distinguishes strict from material implication, there is some X and Y such that X does not strictly imply Y and X does not strictly imply that Y is false (Lewis and Langford 1932, 179). This postulate can be symbolised in the following way: $\exists X\exists Y(\neg(X \Rightarrow Y) \land \neg(X \Rightarrow \neg Y))$, which by definition is equivalent with

$\exists X \exists Y(\neg \Box(X \to Y) \wedge \neg \Box(X \to \neg Y))$. The existence postulate is equivalent with the proposition that there is something contingent ($\exists X \triangledown X$). Let us now prove this. $\exists X \triangledown X$ is, by definition, equivalent with the claim that there is something that is possibly true and possibly false ($\exists X(\Diamond X \wedge \Diamond \neg X)$). So, to prove that the existence postulate is equivalent with the proposition that there is something contingent, we prove that the existence postulate is equivalent with $\exists X(\Diamond X \wedge \Diamond \neg X)$. I show one direction and leave the other to the reader.

$\exists X(\Diamond X \wedge \Diamond \neg X)$ is implied by the existence postulate.

(1) $\neg(\exists X \exists Y(\neg \Box(X \to Y) \wedge \neg \Box(X \to \neg Y)) \to \exists X(\Diamond X \wedge \Diamond \neg X))$, 0
(2) $\exists X \exists Y(\neg \Box(X \to Y) \wedge \neg \Box(X \to \neg Y))$, 0 [1, $\neg \to$]
(3) $\neg \exists X(\Diamond X \wedge \Diamond \neg X)$, 0 [1, $\neg \to$]
(4) $\forall X \neg(\Diamond X \wedge \Diamond \neg X)$, 0 [3, $\neg \exists$]
(5) $\exists Y(\neg \Box(Z \to Y) \wedge \neg \Box(Z \to \neg Y))$, 0 [2, $\exists$ Z/X]
(6) $\neg \Box(Z \to W) \wedge \neg \Box(Z \to \neg W)$, 0 [5, $\exists$ W/Y]
(7) $\neg \Box(Z \to W)$, 0 [6, $\wedge$]
(8) $\neg \Box(Z \to \neg W)$, 0 [6, $\wedge$]
(9) $\neg(\Diamond(Z \to W) \wedge \Diamond \neg(Z \to W))$, 0 [4, $\forall$ $(Z \to W)/X$]

$\swarrow$	$\searrow$
(10) $\neg \Diamond(Z \to W)$, 0 [9, $\neg \wedge$]	(11) $\neg \Diamond \neg(Z \to W)$, 0 [9, $\neg \wedge$]
(12) $\Box \neg(Z \to W)$, 0 [10, $\neg \Diamond$]	(13) $\Box \neg\neg(Z \to W)$, 0 [11, $\neg \Diamond$]
(14) $\Diamond \neg(Z \to \neg W)$, 0 [8, $\neg \Box$]	(15) $\Diamond \neg(Z \to W)$, 0 [7, $\neg \Box$]
(16) $0r1$ [14, $\Diamond$]	(17) $0r1$ [15, $\Diamond$]
(18) $\neg(Z \to \neg W)$, 1 [14, $\Diamond$]	(19) $\neg(Z \to W)$, 1 [15, $\Diamond$]
(20) Z, 1 [18, $\neg \to$]	(21) $\neg\neg(Z \to W)$, 1 [13, 17, $\Box$]
(22) $\neg\neg W$, 1 [18, $\neg \to$]	(23) $*$ [19, 21]
(24) $\neg(Z \to W)$, 1 [12, 16, $\Box$]	
(25) Z, 1 [24, $\neg \to$]	
(26) $\neg W$, 1 [24, $\neg \to$]	
(27) $*$ [22, 26]	

The existence postulate is also equivalent with the proposition that there is an X and a Y such that X is both consistent with Y and with the negation of Y, $\exists X \exists Y((X \circ Y) \wedge (X \circ \neg Y))$, which by definition is equivalent with $\exists X \exists Y(\Diamond(X \wedge Y) \wedge \Diamond(X \wedge \neg Y))$. All these equivalences hold in every system in this paper.

In all of our systems we can prove the following sentences: if something is contingent, then not everything is non-contingent ($\exists X \triangledown X \to \neg \forall X \blacktriangledown X$), if not everything is non-contingent, then something is contingent ($\neg \forall X \blacktriangledown X \to \exists X \triangledown X$), something is contingent iff not everything is non-contingent ($\exists X \triangledown X \leftrightarrow \neg \forall X \blacktriangledown X$), if everything is contingent, then nothing is non-contingent

($\forall X \triangledown X \rightarrow \neg\exists X \blacktriangledown X$), if nothing is non-contingent, then everything is contingent ($\neg\exists X \blacktriangledown X \rightarrow \forall X \triangledown X$), everything is contingent iff nothing is non-contingent ($\forall X \triangledown X \leftrightarrow \neg\exists X \blacktriangledown X$), everything is non-contingent iff nothing is contingent ($\forall X \blacktriangledown X \leftrightarrow \neg\exists X \triangledown X$), and something is non-contingent iff not everything is contingent ($\exists X \blacktriangledown X \leftrightarrow \neg\forall X \triangledown X$). In every system that includes $T-F$, we can prove that everything is non-contingent, that is, we can show that $\forall X \blacktriangledown X$ is valid. $\forall X \blacktriangledown X$ is equivalent with the negation of the proposition that there is something contingent. So, $T-F$ is incompatible with the existence postulate. Let us now show that $\forall X \blacktriangledown X$ is a theorem in every $T-F$-system. $\forall X \blacktriangledown X$ is equivalent with $\forall X \neg \triangledown X$, which is equivalent with $\forall X \neg(\Diamond X \wedge \Diamond \neg X)$, which is equivalent with $\forall X(\Box X \vee \blacklozenge X)$ (everything is either necessary or impossible), which is equivalent with $\forall X(\Box X \vee \Box \neg X)$ (everything is necessarily true or necessarily false). So, to prove that $\forall X \blacktriangledown X$ is a theorem in every $T-F$-system, we show that $\forall X(\Box X \vee \Box \neg X)$ is a theorem in every $T-F$-system.

$\forall X(\Box X \vee \Box \neg X)$. Everything is non-contingent.
(1) $\neg\forall X(\Box X \vee \Box \neg X)$, 0
(2) $\exists X \neg(\Box X \vee \Box \neg X)$, 0 [1, $\neg\forall$]
(3) $\neg(\Box P \vee \Box \neg P)$, 0 [2, $\exists$]
(4) $\neg \Box P$, 0 [3, $\neg\vee$]
(5) $\neg \Box \neg P$, 0 [3, $\neg\vee$]
(6) $\Diamond \neg P$, 0 [4, $\neg \Box$]
(7) $\Diamond \neg\neg P$, 0 [5, $\neg \Box$]
(8) $0r1$ [7, $\Diamond$]
(9) $\neg\neg P$, 1 [7, $\Diamond$]
(10) P, 1 [9, $\neg\neg$]
(11) $0r2$ [6, $\Diamond$]
(12) $\neg P$, 2 [6, $\Diamond$]
(13) $1 = 2$ [8, 11, $T-F$]
(14) P, 2 [10, 13, $T-Ii$]
(15) $*$ [12, 14]

Note that $\exists X(\Diamond X \wedge \Diamond \neg X)$ is true in a possible world ω only if there are at least two possible worlds that are accessible from ω, one in which X is true and one in which X is false (for some X). But according to $C-F$, every possible world can see at most one possible world.

Here are some other theorems that include the contingency and noncontingency operators and that can be proved in every system: $\forall X(\triangledown X \leftrightarrow \triangledown \neg X)$ (for every X: X is contingent iff the negation of X is contingent), $\forall X(\blacktriangledown X \leftrightarrow$

$\blacktriangledown \neg X$) (for every X: X is non-contingent iff the negation of X is non-contingent), $\forall X(\Box X \vee \triangledown X \vee \blacklozenge X)$ (everything is either necessary, contingent or impossible), nothing is both necessary and contingent ($\neg\exists X(\Box X \wedge \triangledown X)$), and nothing is both contingent and impossible ($\neg\exists X(\triangledown X \wedge \blacklozenge X)$).

We cannot show that nothing is both necessary and impossible in every system in this paper, but $\neg\exists X(\Box X \wedge \blacklozenge X)$ is a theorem in every system that includes T or D. If there is something that is necessary and impossible, then everything is necessary and impossible; $\exists X(\Box X \wedge \blacklozenge X) \rightarrow \forall X(\Box X \wedge \blacklozenge X)$ is a theorem in every system in this paper.

If everything is either necessary, contingent or impossible and not everything is necessary or impossible, then obviously something is contingent. And if everything is either necessary, contingent or impossible and something is contingent, then obviously not everything is necessary or impossible.

We have shown that something is necessary, that something is impossible and that something is contingent, given the existence postulate. Since nothing is both necessary and contingent, nothing is both contingent and impossible and nothing is both necessary and impossible (given T or D), we can see that there are at least three non-equivalent propositions if we assume the existence postulate (and our underlying logic includes T or D). However, given the existence postulate we can prove something stronger, namely that there are at least four distinct propositions: something that is necessary, something that is true but not necessary, something that is false but not impossible and something that is impossible. Before we prove this, let us first note that $\neg\exists X\exists Y\exists Z(\neg(X \Leftrightarrow Y) \wedge \neg(X \Leftrightarrow Z) \wedge \neg(Y \Leftrightarrow Z))$ is derivable from $\forall X(\Box X \vee \neg \Diamond X)$ in every system (it is left to the reader to verify this claim). If everything is non-contingent, there are no more than two distinct propositions. Recall that the distinctions between what is true, necessary and possible collapse and that the distinctions between what is false, impossible and possibly false collapse if we assume $C-F$ (and $C-T$) and that all necessary propositions are necessarily equivalent and that all impossible propositions are necessarily equivalent. Since we also have observed that there are at least two distinct propositions, we can conclude that there are exactly two distinct propositions according to some systems.

Assume that the existence postulate holds. Then there are at least four distinct propositions (at least in every T-system). In other words, we have that $\exists X\exists Y\exists Z\exists W(\neg\Box(X \leftrightarrow Y) \wedge \neg\Box(X \leftrightarrow Z) \wedge \neg\Box(X \leftrightarrow W) \wedge \neg\Box(Y \leftrightarrow Z) \wedge \neg\Box(Y \leftrightarrow W) \wedge \neg\Box(Z \leftrightarrow W))$ is derivable from $\exists X\exists Y(\neg\Box(X \rightarrow$

$Y) \wedge \neg\Box(X \rightarrow \neg Y))$. To prove this claim, we will first establish that there is something that is true but not necessarily true ($\exists X(X \wedge \neg\Box X)$), or —in other words— that there is something that is true but possibly false ($\exists X(X \wedge \Diamond\neg X)$), and that there is something that is false but not impossible ($\exists X(\neg X \wedge \neg\blacklozenge X)$), or —in other words— that there is something that is false but possibly true ($\exists X(\neg X \wedge \Diamond X)$). I will prove the first proposition and leave the second to the reader. Since we have shown that the existence postulate is equivalent to the proposition that there is something contingent, we have established the desired result if we can show that $\exists X(X \wedge \neg\Box X)$ is derivable from $\exists X(\Diamond X \wedge \Diamond\neg X)$. Let us now prove this.

$\exists X(X \wedge \neg\Box X)$ is derivable from $\exists X(\Diamond X \wedge \Diamond\neg X)$.

(1) $\exists X(\Diamond X \wedge \Diamond\neg X)$, 0
(2) $\neg\exists X(X \wedge \neg\Box X)$, 0
(3) $\forall X\neg(X \wedge \neg\Box X)$, 0 [2, $\neg\exists$]
(4) $\Diamond Y \wedge \Diamond\neg Y$, 0 [1, $\exists$]
(5) $\Diamond Y$, 0 [4, $\wedge$]
(6) $\Diamond\neg Y$, 0 [4, $\wedge$]
(7) $\neg(Y \wedge \neg\Box Y)$, 0 [3, $\forall$]
(8) $\neg(\neg Y \wedge \neg\Box\neg Y)$, 0 [3, $\forall$]

↙ (9) $\neg Y$, 0 [7, $\neg\wedge$]		↘ (10) $\neg\neg\Box Y$, 0 [7, $\neg\wedge$]
		(11) $\Box Y$, 0 [10, $\neg\neg$]
↙ (12) $\neg\neg Y$, 0 [8, $\neg\wedge$]	↘ (13) $\neg\neg\Box\neg Y$, 0 [8, $\neg\wedge$]	(14) $0r1$ [6, $\Diamond$]
(15) $*$ [9, 12]	(16) $\Box\neg Y$, 0 [13, $\neg\neg$]	(17) $\neg Y$, 1 [6, $\Diamond$]
	(18) $0r1$ [5, $\Diamond$]	(19) Y, 1 [11, 14, $\Box$]
	(20) Y, 1 [5, $\Diamond$]	(21) $*$ [17, 19]
	(22) $\neg Y$, 1 [16, 18, $\Box$]	
	(23) $*$ [20, 22]	

From $\exists X(X \wedge \neg\Box X)$ we can derive the proposition that there is something contingently true (true and contingent) in every system that includes D (or T), that is, the following formula is derivable in all D- and T-systems: $\exists X(X \wedge \nabla X)$, which is equivalent with $\exists X(X \wedge (\Diamond X \wedge \Diamond\neg X))$. Likewise, $\exists X(\neg X \wedge \nabla X)$ (which is equivalent with $\exists X(\neg X \wedge (\Diamond X \wedge \Diamond\neg X))$ is derivable from $\exists X(\neg X \wedge \neg\blacklozenge X)$ in all D- and T-systems. According to $\exists X(\neg X \wedge \nabla X)$, there is something contingently false (false and contingent). Hence, there is something that is necessary, there is something that is true but not necessary, there is something that is false but not impossible, and there is something that is impossible. Now, we can use these propositions as premises in an argument for our conclusion, that is, we can show that $\exists X \exists Y \exists Z \exists W(\neg\Box(X \leftrightarrow Y) \wedge \neg\Box(X \leftrightarrow Z) \wedge \neg\Box(X \leftrightarrow W) \wedge \neg\Box(Y \leftrightarrow Z) \wedge \neg\Box(Y \leftrightarrow W) \wedge \neg\Box(Z \leftrightarrow W))$ is derivable from $\exists X \Box X$, $\exists X(X \wedge \neg\Box X)$, $\exists X(\neg X \wedge \Diamond X)$ and $\exists X \Box \neg X$. Since all these premises are derivable from the existence postulate, it follows that the proposition that there are at least four distinct propositions is derivable from the existence postulate. It is left to the reader to verify these claims (assume that the system includes T).

Let us conclude this section by mentioning some other things that can be derived from the existence postulate (in systems with T). Given the existence postulate, we can prove that there are at least three distinct propositions which are possible, that is, the following formula is derivable from the existence postulate: $\exists X \exists Y \exists Z((\Diamond X \wedge \Diamond Y \wedge \Diamond Z) \wedge \neg\Box(X \leftrightarrow Y) \wedge \neg\Box(X \leftrightarrow Z) \wedge \neg\Box(Y \leftrightarrow Z))$. To prove this proposition we can use the following premises: $\exists X \Box X$, $\exists X(X \wedge \neg\Box X)$ and $\exists X(\neg X \wedge \Diamond X)$. Furthermore, given the existence postulate, we can prove that there are at least three distinct propositions which are possibly false (not necessarily true), that is, the following formula is derivable from the existence postulate: $\exists X \exists Y \exists Z((\Diamond\neg X \wedge \Diamond\neg Y \wedge \Diamond\neg Z) \wedge \neg\Box(X \leftrightarrow Y) \wedge \neg\Box(X \leftrightarrow Z) \wedge \neg\Box(Y \leftrightarrow Z))$. To prove this proposition we can use the following premises: $\exists X(X \wedge \neg\Box X)$, $\exists X(\neg X \wedge \Diamond X)$ and $\exists X \Box \neg X$. This might suggest that there are at least six distinct propositions. However, we cannot show this.

One of the main reasons behind the existence postulate for Lewis was that he wanted a postulate that categorically distinguished the system(s) of strict implication from that of material implication. And given the existence postulate we can show that strict implication does not collapse into material implication. In other words, if we accept the existence postulate we can show that there is some X and some Y such that X materially implies Y even though X does not necessarily imply Y, that is $\exists X \exists Y((X \to Y) \wedge \neg\Box(X \to Y))$ is derivable from $\exists X \exists Y(\neg\Box(X \to Y) \wedge \neg\Box(X \to \neg Y))$. Since we have shown that the existence postulate is equivalent with $\exists X(\Diamond X \wedge \Diamond\neg X)$, we will use this formula in our derivation.

$\exists X \exists Y((X \to Y) \wedge \neg\Box(X \to Y))$ is derivable from $\exists X(\Diamond X \wedge \Diamond\neg X)$.

(1) $\exists X(\Diamond X \wedge \Diamond\neg X)$, 0

(2) $\neg\exists X \exists Y((X \to Y) \wedge \neg\Box(X \to Y))$, 0

(3) $\Diamond Z \wedge \Diamond\neg Z$, 0 [1, $\exists$]

(4) $\Diamond Z$, 0 [3, $\wedge$]

(5) $\Diamond\neg Z$, 0 [3, $\wedge$]

(6) $\forall X \neg \exists Y((X \to Y) \wedge \neg \Box (X \to Y))$, 0 [2, $\neg\exists$]
(7) $\neg \exists Y((\neg Z \to Y) \wedge \neg \Box (\neg Z \to Y))$, 0 [6, $\forall$]
(8) $\forall Y \neg((\neg Z \to Y) \wedge \neg \Box (\neg Z \to Y))$, 0 [7, $\neg\exists$]
(9) $\neg((\neg Z \to Z) \wedge \neg \Box (\neg Z \to Z))$, 0 [8, $\forall$]
(10) $\neg \exists Y((Z \to Y) \wedge \neg \Box (Z \to Y))$, 0 [6, $\forall$]
(11) $\forall Y \neg((Z \to Y) \wedge \neg \Box (Z \to Y))$, 0 [10, $\neg\exists$]
(12) $\neg((Z \to \neg Z) \wedge \neg \Box (Z \to \neg Z))$, 0 [11, $\forall$]

↙ ↘

Left branch:

(13) $\neg(\neg Z \to Z)$, 0 [9, $\neg\wedge$]
(15) $\neg Z$, 0 [13, $\neg \to$]
(17) $\neg Z$, 0 [13, $\neg \to$]

↙ ↘

(20) $\neg(Z \to \neg Z)$, 0
(23) Z, 0 [20, $\neg \to$]
(26) $\neg\neg Z$, 0 [20, $\neg \to$]
(29) $*$ [17, 26]

(21) $\neg\neg \Box (Z \to \neg Z)$, 0
(24) $\Box (Z \to \neg Z)$, 0
(27) $0r1$ [4, $\Diamond$]
(30) Z, 1 [4, $\Diamond$]
(31) $Z \to \neg Z$, 1 [24, 27, $\Box$]
(32) $\neg Z$, 1 [30, 31, MP]
(33) $*$ [30, 32]

Right branch:

(14) $\neg\neg \Box (\neg Z \to Z)$, 0 [9, $\neg\wedge$]
(16) $\Box (\neg Z \to Z)$, 0 [14, $\neg\neg$]
(18) $0r1$ [5, $\Diamond$]
(19) $\neg Z$, 1 [5, $\Diamond$]
(22) $\neg Z \to Z$, 1 [16, 18, $\Box$]
(25) Z, 1 [19, 22, MP]
(28) $*$ [19, 25]

5 Soundness and completeness

All the normal modal tableau systems (without propositional quantifiers) in this paper are sound and complete with respect to their semantics. In this section, I will go through the main steps in the soundness and completeness proofs. Then, I will show how to modify the proofs to establish that the extended systems are sound. It is an open question whether or not the extended systems are complete.[4]

Let $S = T - X_1, \dots, T - X_n$ be a normal modal tableau system as defined above (see Section 4.1). Then we shall say that the class of models, $\mathfrak{M}$, corresponds to S iff $\mathfrak{M} = \mathfrak{M}(C - X_1, \dots, C - X_n)$.

S is sound with respect to $\mathfrak{M}$ iff $\Gamma \vdash_S A$ entails $\mathfrak{M}, \Gamma \Vdash A$. S is complete with respect to $\mathfrak{M}$ iff $\mathfrak{M}, \Gamma \Vdash A$ entails $\Gamma \vdash_S A$.

[4] The soundness and completeness proofs for the normal modal tableau systems in this section are similar to some soundness and completeness proofs that can be found in Priest (2008), chapters 2 and 3.

5.1 Soundness theorems

Let $\mathcal{M} = \langle \mathfrak{W}, \mathfrak{R}, \mathfrak{v} \rangle$ be a model and let $\mathfrak{B}$ be a branch in a tableau. Then $\mathfrak{B}$ is satisfiable in $\mathcal{M}$ iff there is a function f from $0, 1, 2, \ldots$ to $\mathfrak{W}$ such that (i) A is true in $f(i)$ in $\mathcal{M}$, for every node A, i on $\mathfrak{B}$, (ii) if irj is on $\mathfrak{B}$, then $\mathfrak{R}f(i)f(j)$ in $\mathcal{M}$, (iii) if $i = j$ is on $\mathfrak{B}$, then $f(i)$ is $f(j)$. If f satisfies these conditions, we say that f shows that $\mathfrak{B}$ is satisfiable in $\mathcal{M}$.

(Soundness lemma I). If the branch $\mathfrak{B}$ is satisfiable in the model $\mathcal{M}$ and a tableau rule is applied to it, then it produces at least one extension, $\mathfrak{B}'$, such that $\mathfrak{B}'$ is satisfiable in $\mathcal{M}$.
Proof. The proof is by induction on the height of the derivation. Let f be a function that shows that the branch $\mathfrak{B}$ is satisfiable in $\mathcal{M}$.

Connectives. Straightforward.

Modal operators. Suppose that $\Box A, i$ and irj are on $\mathfrak{B}$ and that we apply $(\Box)$ to $\mathfrak{B}$. Then we get an extension, $\mathfrak{B}'$, of $\mathfrak{B}$ that includes A, j. Since $\mathfrak{B}$ is satisfiable in $\mathcal{M}$, $\Box A$ is true in $f(i)$. Furthermore, for any i and j such that irj is on $\mathfrak{B}$, $\mathfrak{R}f(i)f(j)$. Hence, A is true in $f(j)$. Suppose that $\Diamond A, i$ is on $\mathfrak{B}$ and that we apply $(\Diamond)$ to $\mathfrak{B}$. Then we obtain an extension, $\mathfrak{B}'$, of $\mathfrak{B}$ that contains nodes of the form irj and A, j. Since $\mathfrak{B}$ is satisfiable in $\mathcal{M}$, $\Diamond A$ is true in $f(i)$. Hence, for some ω in $\mathfrak{W}$, $\mathfrak{R}f(i)\omega$ and A is true in ω. Let f' be the same as f except that $f'(j) = \omega$. Since f and f' differ only at j, f' shows that $\mathfrak{B}$ is satisfiable in $\mathcal{M}$. Moreover, by definition, $\mathfrak{R}f'(i)f'(j)$ and A is true in $f'(j)$. Hence, f' shows that $\mathfrak{B}'$ is satisfiable in $\mathcal{M}$.

$(\neg \Box)$ and $(\neg \Diamond)$. Similar.

(CUT) and the identity rules. CUT is obviously sound since every sentence is either true or false in a possible world. Suppose that $A(i)$ and $i = j$ are on $\mathfrak{B}$ and that we apply $T - Ii$ to get $A(j)$. Then $f(i) = f(j)$, for $\mathfrak{B}$ is satisfiable in $\mathcal{M}$. If $A(i)$ is B, i, then B is true in $f(i)$. Accordingly, B is true in $f(j)$, as required. The other cases are similar.

Accessibility rules. I will consider one example to illustrate the method. $(T - F)$. Suppose that irj and irk are on $\mathfrak{B}$ and that we apply $(T - F)$ to $\mathfrak{B}$ to get an extension that contains $i = j$. Since $\mathfrak{B}$ is satisfiable in $\mathcal{M}$, $\mathfrak{R}f(i)f(j)$ and $\mathfrak{R}f(i)f(k)$. Hence, $f(j) = f(k)$, for $\mathfrak{R}$ satisfies $(C - F)$. Other steps: similar. □

(Soundness theorem I). Every normal modal system in this paper is sound with respect to its semantics.

Proof. Suppose that B does not follow from Γ in $\mathfrak{M}$, where $\mathfrak{M}$ is the class of models that corresponds to S. Then every premise in Γ is true and the conclusion B false in some world ω in some model $\mathcal{M}$ in $\mathfrak{M}$. Consider an S-tableau whose initial list consists of $A, 0$ for every A in Γ and $\neg B, 0$, where '0' refers to ω. Then the initial list is satisfiable in $\mathfrak{M}$. Every time we apply a rule to this list it produces at least one extension that is satisfiable in $\mathfrak{M}$ (by the Soundness Lemma). Accordingly, we can find a whole branch such that every initial section of this branch is satisfiable in $\mathfrak{M}$. If this branch is closed, then some sentence is both true and false in some possible world in $\mathcal{M}$ in $\mathfrak{M}$. Nevertheless, this is impossible. Hence, the tableau is open. Accordingly, B is not derivable from Γ in S. Consequently, if B is derivable from Γ in S, then B follows from Γ in $\mathfrak{M}$. □

5.2 Completeness theorems

In this section, I will show that all the normal modal tableau systems (without propositional quantifiers) in this paper are complete with respect to their semantics. However, first I will introduce some important concepts.

Intuitively, a complete branch is a branch where every rule that can be applied has been applied and a complete tableau is a tableau where every rule that can be applied has been applied. There can be several different (complete) tableaux for the same sentence or set of sentences, since the tableau rules may be applied in different orders. To produce a complete tableau, we can use the following method (which is usually not the simplest one). (1) For every open branch on the tree, one at a time, start from its root and move towards its tip. Apply any rule that produces something new to the branch. For example, ($\Diamond$) is applied at most once to a node of the form $\Diamond A, i$. We do not apply any rules to a branch that is closed. Some rules may have several possible applications, for example, $\Box$. Then we make all applications at once. (2) When we have done this for all open branches on the tree, we repeat the procedure. $(T-D)$ introduces a new possible world. If a rule introduces a new possible world, it is applied once at the tip of every open branch at the end of every cycle when we have dealt with all nodes. If there is still something to do according to this method, the tableau is incomplete; if not, it is complete.

(Induced Model). Let $\mathfrak{B}$ be an open complete branch of a tableau, let i, j, k, etc. be numbers on $\mathfrak{B}$, and let I be the set of numbers on $\mathfrak{B}$. We shall say that $i \sim j$ just in case $i = j$, or '$i = j$' or '$j = i$' occurs on $\mathfrak{B}$. $\sim$ is an equivalence relation and $[i]$ is the equivalence class of i. The model $\mathcal{M} = \langle \mathfrak{W}, \mathfrak{R}, \mathfrak{v} \rangle$ induced by $\mathfrak{B}$ is defined as follows. $\mathfrak{W} = \left\{\omega_{[i]} : i \in I\right\}$, $\mathfrak{R}\omega_{[i]}\omega_{[j]}$ iff irj occurs on $\mathfrak{B}$. Let P be a propositional variable. Then, if P, i occurs on $\mathfrak{B}$, then P is true in $\omega_{[i]}$, that is, $\omega_{[i]}$ is an element in $\mathfrak{v}(P)$, and if $\neg P, i$ occurs on $\mathfrak{B}$, then P is false in $\omega_{[i]}$, that is, $\omega_{[i]}$ is not an element in $\mathfrak{v}(P)$.

If our tableau system does not include $T - F$, $\sim$ is reduced to identity and $[i] = \{i\}$. Hence, in such systems, we may take $\mathfrak{W}$ to be $\{\omega_i : i \textit{ occurs on } \mathfrak{B}\}$ and dispense with the equivalence classes.

(Completeness Lemma). Let $\mathfrak{B}$ be an open branch in a complete tableau and let $\mathcal{M}$ be a model induced by $\mathfrak{B}$. Then, for every formula A:

(i) if A, i is on $\mathfrak{B}$, then A is true in $\omega_{[i]}$, and

(ii) if $\neg A, i$ is on $\mathfrak{B}$, then A is false in $\omega_{[i]}$.

Proof. The proof is by induction on the complexity of A.

(i) Atomic formulas. The result is true by definition.

Connectives. I will consider two cases to illustrate the method. $(\wedge)$ Suppose A is of the form $B \wedge C$. Then $(\wedge)$ has been applied to $B \wedge C, i$ and we have B, i and C, i on $\mathfrak{B}$. By the induction hypothesis, B is true in $\omega_{[i]}$, and C is true in $\omega_{[i]}$. It follows that $B \wedge C$ is true in $\omega_{[i]}$, as required. $(\vee)$ If A is of the form $B \vee C$, then $(\vee)$ has been applied to $B \vee C, i$. Hence, either B, i or C, i is on $\mathfrak{B}$. By the induction hypothesis, either B is true in $\omega_{[i]}$ or C is true in $\omega_{[i]}$. It follows that $B \vee C$ is true in $\omega_{[i]}$, as required. Other cases. Similar.

Modal operators. $(\Box)$. Suppose that A is of the form $\Box B$ and that $\Box B, i$ is on $\mathfrak{B}$. Then for all j such that irj is on $\mathfrak{B}$, B, j is on $\mathfrak{B}$. By the induction hypothesis, for all $\omega_{[j]}$ such that $\mathfrak{R}\omega_{[i]}\omega_{[j]}$, B is true in $\omega_{[j]}$. Consequently, $\Box B$ is true in $\omega_{[i]}$, as required. $(\Diamond)$ Suppose that A is of the form $\Diamond B$ and that $\Diamond B, i$ is on $\mathfrak{B}$. Then for some j, irj and B, j are on $\mathfrak{B}$. By the induction hypothesis, $\mathfrak{R}\omega_{[i]}\omega_{[j]}$ and B is true in $\omega_{[j]}$. It follows that $\Diamond B$ is true in $\omega_{[i]}$, as required.

(ii) Atomic formulas. The result is true by definition.

Connectives. I will consider one example to illustrate the method. $(\neg\vee)$ Suppose A is of the form $\neg(B \vee C)$. Then $(\neg\vee)$ has been applied to $\neg(B \vee C), i$ and we have $\neg B, i$ and $\neg C, i$ on $\mathfrak{B}$. By the induction hypothesis, B is false in $\omega_{[i]}$, and C is false in $\omega_{[i]}$. It follows that $B \vee C$ is false in $\omega_{[i]}$, as required. Other cases. Similar.

Modal operators. $(\neg\Box)$ A is of the form $\neg\Box B$. Suppose that $\neg\Box B, i$ is on $\mathfrak{B}$. Then $\Diamond\neg B, i$ is on $\mathfrak{B}$. Hence, for some j, irj and $\neg B, j$ are on $\mathfrak{B}$. By the induction hypothesis, $\mathfrak{R}\omega_{[i]}\omega_{[j]}$ and B is false in $\omega_{[j]}$. It follows that $\Box B$ is false in $\omega_{[i]}$, as required. $(\neg\Diamond)$. Similar. □

(Completeness Theorem I). Every normal modal system in this paper is complete with respect to its semantics.

Proof. First we prove that the theorem holds for our weakest system K. Then we extend the theorem to all normal modal systems that are stronger than K. Let $\mathfrak{M}$ be the class of models that corresponds to K, that is, $\mathfrak{M}$ is the class of all models.

Suppose that B is not derivable from Γ in K. Then it is not the case that there is a closed K-tableau whose initial list comprises $A, 0$ for every A in Γ and $\neg B, 0$. Let t be a complete K-tableau whose initial list comprises $A, 0$ for every A in Γ and $\neg B, 0$. Then t is not closed, t is open. Hence, there is at least one open branch in t. Let $\mathfrak{B}$ be an open branch in t. The model induced by $\mathfrak{B}$ makes all the premises in Γ true and B false in $\omega_{[0]}$. Accordingly, it is not the case that B follows from Γ in $\mathfrak{M}$. Consequently, if B follows from Γ in $\mathfrak{M}$, then B is derivable from Γ in K.

To prove that all normal modal extensions of K are complete with respect to their semantics, we have to check that the model induced by the open branch $\mathfrak{B}$ is of the right kind. To do this we first check that this is true for every single semantic condition. Then we combine each of the individual arguments. I will go through one step to illustrate the method.

$C - F$. Suppose that $\mathfrak{R}\omega_{[i]}\omega_{[j]}$ and $\mathfrak{R}\omega_{[i]}\omega_{[k]}$. Then irj and irk occur on $\mathfrak{B}$ [by the definition of an induced model]. Since $\mathfrak{B}$ is complete, $(T - F)$ has been applied and $j = k$ occurs on $\mathfrak{B}$. Hence, $j \sim k$. So, $[j] = [k]$. It follows that $\omega_{[j]} = \omega_{[k]}$, as required [by the definition of an induced model]. □

5.3 Soundness for systems with propositional quantifiers

In the extended systems, a tableau branch can contain propositional quantifiers. Hence, we must modify the soundness proof slightly.

(Soundness lemma II). If the branch $\mathfrak{B}$ is satisfiable in the model $\mathcal{M}$ and a tableau rule is applied to it, then there is a model $\mathcal{M}'$ and an extension of $\mathfrak{B}$, $\mathfrak{B}'$, such that $\mathfrak{B}'$ is satisfiable in $\mathcal{M}'$.

Proof. Most cases are trivial modifications of the steps in the proof for soundness lemma I in Section 5.1 (just let $\mathcal{M}'$ be $\mathcal{M}$). The only really interesting new steps are the steps for the quantifiers.

($\forall$). Straightforward.

($\exists$). Suppose that $\exists XA, i$ is on $\mathfrak{B}$ and that we apply ($\exists$) to $\mathfrak{B}$. Then we get an extension, $\mathfrak{B}'$, of $\mathfrak{B}$ that includes $A[Y/X], i$, where Y is a propositional variable new to the branch. Since $\mathfrak{B}$ is satisfiable in $\mathcal{M}$, $\exists XA$ is true in $f(i)$. Hence, there is some sentence B in $\mathcal{L}$ such that B is substitutable for X in A and $A[B/X]$ is true in $f(i)$. Let $\mathcal{M}' = \langle \mathfrak{W}, \mathfrak{R}, \mathfrak{v}' \rangle$ be like $\mathcal{M}$, except that $\mathfrak{v}'(Y) = \{f(j) : \mathcal{M}', f(j) \Vdash B\}$. Then $A[Y/X]$ is true in $f(i)$ in $\mathcal{M}'$. □

(Soundness theorem II). All augmented systems are sound with respect to their semantics.

Proof. Once soundness lemma II is established, the rest of the proof is an easy modification of the soundness proof for soundness theorem I. □

5.4 An example of a countermodel

In this section, I will consider one example of a countermodel that proves that a certain sentence is not valid and hence that it is not a theorem in a certain system. This will illustrate how one can use countermodels to prove invalidity and 'non-theoremhood'.

The following sentence is a theorem in every (extended) system that includes $T - D$ (or $T - T$): $\forall X(\Box X \to \Diamond X)$. $\forall X(\Box X \to \Diamond X)$ says that everything that is necessary is possible. But this formula is not a theorem in every system. It is, for example, not a theorem in the smallest extended system K_{Ext}. To show this, we will first establish that the sentence is not valid in the class of all models. The following model establishes this: $\mathcal{M} = \langle \mathfrak{W}, \mathfrak{R}, \mathfrak{v} \rangle$, where $\mathfrak{W} = \omega$, $\mathfrak{R}$ is empty, and P is true in ω (or false; it does not matter). Since no possible world is accessible from ω, it is vacuously true that P is true in every possible world that is accessible from ω. Hence, $\Box P$ is true in ω. $\Diamond P$ is true in ω only if P is true in some possible world that is accessible from ω. Since no possible world is accessible from ω, it is not the case that P is true in some possible world that is accessible from ω. Accordingly, $\Diamond P$ is not true in ω. It follows that $\Box P \to \Diamond P$ is false in ω, for the antecedent is true and the consequent is false in ω. $\forall X(\Box X \rightarrow \Diamond X)$ is true in ω only if $(\Box X \rightarrow \Diamond X)[P/X]$ is true in ω. But $(\Box X \rightarrow \Diamond X)[P/X] = (\Box P \rightarrow \Diamond P)$ and $\Box P \rightarrow \Diamond P$ is false in ω. Therefore, $\forall X(\Box X \rightarrow \Diamond X)$ is not true in ω. Consequently, $\forall X(\Box X \rightarrow \Diamond X)$ is not valid in the class of all models. We have shown that the system K (and K_{Ext}) is sound with respect to the class of all models (Sections 5.1 and 5.3). It follows that $\forall X(\Box X \rightarrow \Diamond X)$ is not a theorem in K_{Ext}.

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REPORT

Modal Metaphysics: Issues on the (Im)Possible VII*

On May 30–31, 2019 the Institute of Philosophy of the Slovak Academy of Sciences, the Slovak Philosophical Association, the Department of Logic and Methodology of Sciences at Comenius University and `metaphysics.sk` hosted the seventh Issues on the (Im)Possible Conference. As usual, the conference was hosted by the Slovak Academy of Sciences in Bratislava.

Beside the conference itself, the event accommodated three additional streams: Current Trends in Deontic Logics II, Semantics of Fictional Discourse II and Truth in Time and Open Future Stream II. Given the size as well as the scope of the conference, it was rather unsurprising that the organizers invited more keynote speakers. Namely: Jan Broersen (University of Utrecht, The Netherlands), Gregory Currie (University of York, UK), Peter van Inwagen (University of Notre Dame, USA), Peter Lamarque (University of York, UK) and Scott Shalkowski (University of Leeds, UK).

During two intensive days, the following speakers contributed to the conference: Kirk Lougheed (McMaster University): "Epistemically Possible Worlds and the Counterpossible Objection to the Axiology of Theism" (Commentator: Ethan Brauer); Gonzalo Rodriguez-Pereyra (Oxford University): "Are the Laws of Metaphysics Metaphysically Necessary?"; Yael Loewenstein (University of Houston): "Against the Standard Solution to the Grandfather Paradox (And How Not to Understand Time-Indexed Modals in Contexts with Backwards Causation)" ; Nathan Wildman (Tilburg University / TiLPs): "Necessity by Accident" (Commentator: Felipe Andrés Morales Carbonell); Martin Glazier (University of Hamburg): "What Time is it in Other Possible Worlds?"

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(Commentator: Yael Loewenstein); Michael De (University of Bern): "Truthmakers or Truthmaking Supervenience?" (Commentator: Nathan Wildman); Benjamin Marschall (University of Cambridge): "Carnap's Internal Platonism" (Commentator: Karol Lenart); Michael Bertrand (Auburn University): "Two Concepts of Metaphysical Grounding" (Commentator: Dirk Franken); Jan Heylen & Felipe Morales (KU Leuven): "Circularity and Modality" (Commentator: Anthony Fisher); Augusto Trujillo Werner (University of Malaga): "Metaethics: Aquinas, Hume and Moore"; Dan Marshall (Lingnan University): "Against Linguistic Ersatzism" (Commentator: Michael De); Krzysztof Wójtowicz (University of Warsaw): "The Modal Character of Program Explanations" (Commentator: Mike Bertrand); Peter Marton (Bridgewater State University): "Without Conceivability. (A Moderate Anti-Realist Approach to Possibility, Meaning ... and Zombies)" (Commentator: Giacomo Giannini); Karol Lenart (Jagiellonian University): "Actualism and Haecceitism" (Commentator: Zach Thornton); Vladimir Lobovikov (Ural Federal University): "Analytical Metaphysics of Modalities, and a Formal Epistemology Axiomatic System Based on Not-Normal Modal Logic" ; Riccardo Baratella (Universität Salzburg): "No Chance for the Change Argument A Reply to Stouts The Category of Occurrent Continuants" (Commentator: Benjamin Marschall); Rheanna Trevino (University of Texas at San Antonio): "Actualism and Being: Ontological Commitments and Modal Logic" ; Dirk Franken (University of Mainz): "On Confusions of Ground and Existence" (Commentator: Alessandro Torza); Anthony Fisher (University of Manchester): "David Lewis and the Role of Theoretical Virtues in Metaphysics" (Commentator: Nathan Wildman); Giacomo Giannini (Durham University): "A Crowded World. Dispositionalism and Necessitism" (Commentator: Dan Marshall); Alessandro Torza (National Autonomous University of Mexico): "Ground and Modality" (Commentator: Martin Glazier); Ethan Brauer (Ohio State University): "Metaphysical Nihilism and Modal Logic" (Commentator: Krzysztof Wójtowicz); Zach Thornton (University of North Carolina at Chapel Hill): "Distinctness as Possible Difference" (Commentator: Peter Marton); Michael Wallner & Anand Jayprakash Vaidya (University of Graz & San Jose State University): "The Structure of Essentialist Explanations of Necessity" (Commentator: Riccardo Baratella).

Streams were represented by the following speakers respectively: Current Trends in Deontic Logics II: Roberto Ciuni (Department FISPPA): "Information based Oughts and their Interaction with Knowledge and Beliefs"; Tereza

Novotná (Brno University): "Network Analysis in Law"; Meha Mishra (Indian Institute of Technology Kanpur): "Tolerating Inconsistencies: A Study of Logic of Moral Conflicts"; Igor Sedlár (Czech Academy of Sciences): "Hyperintesional Deontic Logic" and Daniela Glavaničová & Matteo Pascucci (Slovak Academy of Sciences and Comenius University in Bratislava & Slovak Academy of Sciences and TU Wien): "Defining Responsibility".

Semantics of Fictional Discourse II: Marco Hausmann (Ludwig Maximilians Universität München): "Fictional Realism and Negative Existentials: Why Kripkes Proposal Fails"; Manuel Rebuschi (Université de Lorraine): "IF Modal Logic for Fictions"; Fredrik Stjernberg (Linkoping University): "The No-name Theory of Fictional Names" and Elisa Paganini (Università degli Studi di Milano): "Fictional Knowledge".

Truth in Time and Open Future II: Daniel Steele (University of Dallas): "The Indeterminacies of Future Contingent Propositions"; Vincent Grandjean (University of Neuchatel): "A Model for the Open Future" ; Andrew Cortens (Boise State University): "On the Metaphysical Necessity of the Past"; Elton Marques (University of Rio de Janeiro): "What is Fatalism?"; Jacek Wawer (Jagiellonian University in Krakow): "Ockhamism without Molinism" and Roberto Ciuni & Carlo Proietti (University of Padua & University of Amsterdam): "Postsemantics and the Future Contingents Problem".

Of course, unless such a line-up were present, the conference would not be such a success. On behalf of the organizers, I would like to thank again the speakers for extending the bounds of impossibility.

Martin Vacek

PHILOSOPHY OF LANGUAGE: PROPOSITIONS, REFERENCE AND MEANING

November 28-29, 2019 (Bratislava, SLOVAKIA)

SCOTT SOAMES

(University of Southern California)

We invite submissions for a 30-minute presentation followed by 20-minute discussion. Areas of interest include any aspect of the philosophy of language, and especially Scott Soames's work.

An abstract of approximately 500 words should be prepared for blind review and include a cover page with the full name, title, institution and contact information. Papers can be submitted in pdf or doc(x) and should be sent to **martinvacekphilosophy@gmail.com**.

Deadline for submission: July 31, 2019

Notification of acceptance: August 31, 2019

General inquiries: Martin Vacek
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